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On the geometric meaning of the non-abelian Reidemeister torsion of cusped hyperbolic 3-manifolds

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Abstract

The non-abelian Reidemeister torsion is a numerical invariant of cusped hyperbolic 3-manifolds defined by J. Porti [46] in terms of the adjoint holonomy representation of the hyperbolic structure. We develop a geometric approach to the definition and computation of the torsion using infinitesimal isometries. For manifolds carrying positively oriented geometric ideal triangulations, we establish a fundamental relationship between the derivatives of Thurston's gluing equations and the cohomology of the sheaf of infinitesimal isometries. Using these results, we obtain a partial confirmation of the '1-loop Conjecture' of Dimofte and Garoufalidis [11] which expresses the non-abelian torsion in terms of the combinatorics of the gluing equations. In this way, we reduce the Conjecture to a certain normalization property of the Reidemeister torsion of free groups.

Introduction

Background and motivation

The Mostow–Prasad Rigidity Theorem [4, 29] says that if a smooth manifold M of dimension ≥ 3 admits a complete hyperbolic structure of finite volume, then this hyperbolic structure is unique, and determined in fact by the homotopy type of M alone. As a consequence of the Rigidity Theorem, every property of this geometry is automatically a topological invariant of M. The simplest such invariant is the hyperbolic volume, $\mathrm{Vol}(M) = \int_M \mathrm{dvol}$ where dvol is the hyperbolic volume form on M.

For instance, suppose that $K \subset S^3$ is a knot or a link in the three-dimensional sphere. Very often, the complement $M := S^3 \setminus K$ admits a complete hyperbolic structure of finite volume, but it is highly unclear how $\mathrm{Vol}(S^3 \setminus K)$ relates to other isotopy invariants of K. This is especially true of the so-called 'quantum invariants' of knots and links, most of which are defined in terms of planar diagrams.

A long-standing open problem posed by Rinat Kashaev [30] and later reformulated by Hitoshi Murakami and Jun Murakami [39] is whether

$$\mathrm{Vol}(S^3 \setminus K) = 2\pi \lim_{n\to\infty} \frac{\log |J_n(K, e^{2\pi i/n})|}{n},$$

where $J_n(K, q) \in \mathbb{Z}[q, q^{-1}]$ is the nth *coloured Jones polynomial* of K. Known as the *Volume Conjecture*, the above statement also has a complexified version in which the volume is replaced by the *complex volume* $\mathrm{Vol}(M) + i\mathrm{CS}(M)$, where $\mathrm{CS}(M)$ is the Chern–Simons invariant of M [41]. From the point of view of physics, the Volume Conjecture relates the world of Riemannian geometry (the foundation of Einstein's General Relativity) to the mathematical models pertinent to Quantum Mechanics. The left-hand side is 'continuous' and 'classical', the right-hand side 'combinatorial' and 'quantum'.

There exists a far-reaching generalization of the Volume Conjecture [1, Conjecture 1–1], which in particular predicts that the complex volume is merely the first coefficient in an infinite power series describing the $n \to \infty$ asymptotics of the coloured Jones polynomials to all orders. In addition to the original prediction of the Volume Conjecture, it is expected [22] that the second (sub-leading) coefficient in this asymptotic expansion should be related to another 'classical' invariant, the *non-abelian Reidemeister torsion*.

There are two flavours of Reidemeister torsion: the *combinatorial torsion,* which dates back to the work of Reidemeister and Franz, and the *analytic torsion,* which was first defined by Ray and Singer [48]. Despite its theoretical appeal, the analytic torsion is almost impossible to compute in practice. In this work, we focus exclusively on combinatorial torsion.

The combinatorial torsion is usually defined in terms of a linear representation of the fundamental group of a manifold and can be computed effectively by manipulating bases of a finite-dimensional twisted cellular cochain complex (see Section 3.1). For closed manifolds and orthogonal representations, the combinatorial torsion is equal to the analytic torsion by a theorem proved independently by J. Cheeger and W. Müller [38]. Since the combinatorial torsion is a combinatorial invariant of a CW-complex, its topological invariance is limited to low dimension, where the Hauptvermutung holds. In the case of a finite-volume hyperbolic 3-manifold, the combinatorial Reidemeister torsion relevant to the Generalized Volume Conjecture is the invariant constructed by Joan Porti in [46].

In the recent paper [11], Tudor Dimofte and Stavros Garoufalidis describe a method of using Thurston's gluing equations to compute what they believe to be the correct coefficients of the asymptotic expansion of the Kashaev invariant. Their computation can be carried out with the help of Feynman diagrams with an increasing number of loops, yielding an asymptotic expansion

$$\mathcal{Z}_M(\hbar) = \exp\Big(S_{M,0}\hbar^{-1} - \tfrac{3}{2}\log\hbar + S_{M,1} + S_{M,2}\hbar + S_{M,3}\hbar^2 + \cdots\Big),\ \hbar = \frac{2\pi i}{n}. \quad \text{(eq. 1–3 of [11])}$$

The 'm-loop coefficients' $S_{M,m} \in \mathbb{C}$ are conjectured to be topological invariants of M for all m, but at the time of writing this is only known for $m = 0$. By construction, $S_{M,0}$ is essentially equal to the complex volume of M, as required by the Volume Conjecture.

From our point of view, the most interesting result of [11] is the derivation of a closed formula for the 1-loop coefficient $S_{M,1}$. Conveniently, this formula makes sense not only at the unique complete hyperbolic structure on M, but also in a neighbourhood of it, consisting generically of incomplete hyperbolic structures. The Generalized Volume Conjecture postulates that this explicit expression for $S_{M,1}$ should produce an equally explicit formula for the adjoint non-abelian Reidemeister torsion $\mathbb{T}_{\mathrm{Ad}}$, which Dimofte and Garoufalidis wrote down as

$$\mathbb{T}_{\mathrm{Ad}}(S^3 \setminus K, \mu) \stackrel{?}{=} \pm\tfrac{1}{2}\det\Big(\mathbf{A}\,\mathrm{diag}\,(z'') + \mathbf{B}\,\mathrm{diag}\,(z)^{-1}\Big) \times \prod_{j=1}^{N} z_j^{f_j''} z_j''^{-f_j}, \tag{I-1}$$

where μ is the meridian of the hyperbolic knot K. The above formula, called the 1*-loop Conjecture*, assumes that the knot complement $M = S^3 \setminus K$ admits a geometric ideal triangulation into N tetrahedra. The matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{N\times N}(\mathbb{Z})$ are then defined combinatorially in terms of the incidence numbers of the triangulation and of the chosen meridian path, by the way of Neumann–Zagier [40]. The complex numbers z with various decorations refer to the positively oriented geometric solution of Thurston's gluing equations defining either the complete structure on M or a small deformation of it, whereas the integers f with decorations are a discrete 'shadow solution' of logarithmic gluing equations[1].

The left-hand side of the conjectural formula (I-1) is the adjoint non-abelian Reidemeister torsion $\mathbb{T}_{\mathrm{Ad}}(M, \mu) \in \mathbb{C}_\times/\{\pm 1\}$ of Porti [46], which we introduce in Section 3.5. This invariant is defined as the algebraic (Reidemeister–Franz) torsion of a twisted cochain complex of M with coefficients in the complex Lie algebra $\mathfrak{sl}_2\mathbb{C}$. The twisting comes from the adjoint action of the fundamental group of M via the holonomy representation of a hyperbolic structure. Due to the non-compactness of M, this twisted cochain complex has non-vanishing cohomology; as per Porti's construction, an unambiguous definition of the non-abelian torsion requires the choice of

[1]We discuss Thurston's gluing equations in Section 1.2 and state the 1-loop Conjecture in full detail in Section 5.1.2.

an element of $H_1(\partial M;\mathbb{Z})$, here given by the homology class of the meridian μ. Note that Porti's original definition used twisted homology, but our dual approach using cohomology is completely equivalent [53].

The right-hand side of the conjectural formula is a mysterious algebraic expression involving some geometric information about M. This expression has its origins in the quantization procedure described in [11, Appendix C]. While this procedure itself is not fully rigorous mathematically, the final expression on the right-hand side of (I-1) is well defined. Remarkably, the authors of the 1-loop Conjecture have numerically confirmed its validity in more than 50 000 cases to within 1000 decimal digits [11]. However, no direct theoretical insights into the conjecture have been provided thus far.

Goals of this thesis

The motivating goal of this thesis is to study the 1-loop Conjecture (I-1). This requires relating the algebraic constructions behind the non-abelian torsion to the geometric data pertinent to the combinatorial definition of the 1-loop invariant of Dimofte–Garoufalidis [11]. It turns out that the necessary link is provided by complex analysis.

Since only the conjugacy class of a representation affects the value of the twisted torsion, we wish to apply the philosophy of analytic torsion to the computation of combinatorial torsion. More precisely, we think of the combinatorial torsion as of an intrinsic invariant of a *local system* or, equivalently, of a *flat vector bundle*. For instance, given a local system,[2] we can express the combinatorial Reidemeister torsion in terms of a Čech cochain complex associated to a good open cover. It is thus possible to calculate the twisted combinatorial torsion without writing down any explicit representation of the fundamental group of the topological space in question. We find this approach better suited to the study of triangulated 3-manifolds, in which case we can describe the monodromy of the local system as a representation of a certain equivalent subgroupoid of the fundamental groupoid. This construction is detailed in the largely expository Section 2.3.

An interesting aspect of the expository part of this thesis is our novel description of the holonomy representation of a hyperbolic structure. The *sheaf of germs of developing maps* plays a central role in this description, similar in flavour to the ideas of Guruprasad and Haefliger [23] and related to the work of Fock and Goncharov [17].

From the perspective of the 1-loop Conjecture, the most important application of this philosophy is to the study of the local system defined by the adjoint of the monodromy representation of the sheaf of germs of developing maps on a hyperbolic 3-manifold M. By a result of Matsushima–Murakami [34, Theorem 8.1], this adjoint local system is isomorphic to the sheaf $\mathscr{K}$ of germs of Killing vector fields on M. Analytically, the monodromy of the sheaf $\mathscr{K}$ can be understood as analytic prolongation of germs of Killing fields on M, as studied by K. Nomizu [43].

Since Killing vector fields are interpreted geometrically as infinitesimal isometries of M, we are able to relate the adjoint torsion to the deformation theory of hyperbolic structures. By using elementary methods and capitalizing on earlier results of Thurston [50], Neumann–Zagier [40] and E.-Y. Choi [9], we are able to build Thurston's gluing equations into our computation of the torsion and ultimately to demystify the 1-loop invariant.

[2] By a *local system* we understand a locally constant sheaf of modules over a commutative ring; cf. Definition 1.1.3.

Overview of results

Cohomological interpretation of Thurston's gluing equations

The core results of this thesis concern the relationship between the hyperbolic gluing equations on an ideal triangulation and the cohomology of the sheaf of germs of Killing vector fields. Suppose that $\mathcal{T}$ is a a geometric, positively oriented ideal triangulation of a hyperbolic 3-manifold M with rank two cusps. Denote by $k > 0$ the number of cusps and by N the number of ideal tetrahedra of $\mathcal{T}$. Following Eun-Young Choi [9], we may write the edge consistency equations as a single equation $g(z) = 1$, where g is the gluing map

$$g\colon \mathbb{C}_\times^N \to \mathbb{C}_\times^N, \qquad g(z_1, \ldots, z_N) = \Big(\prod_{j=1}^N z_j{}^{G_{ij}} z_j'{}^{G'_{ij}} z_j''{}^{G''_{ij}}\Big)_{i=1}^N,$$

the exponents being the incidence numbers of the edges of the ideal tetrahedra to the edges of $\mathcal{T}$. The *gluing variety* $\mathcal{V}_\mathcal{T}$ is then the solution space $g^{-1}(1, \ldots, 1)$. Assume that the solution $z_* \in \mathcal{V}_\mathcal{T}$ corresponding to the unique complete structure is positively oriented, i.e., all shape parameters have positive imaginary parts. Thurston [50] proved that z_* admits a neighbourhood in $\mathcal{V}_\mathcal{T}$ which parametrizes the deformed, not necessarily complete hyperbolic structures on M. The 'log-parameters', defined as the complex lengths of certain fixed homotopy classes of peripheral curves, provide local coordinates on $\mathcal{V}_\mathcal{T}$ near z_*. Fixing a system of such peripheral curves, we thus obtain a neighbourhood U of $0 \in \mathbb{C}^k$ together with a biholomorphic map $y\colon U \to \mathcal{V}_\mathcal{T}$ such that $y(0) = z_*$ [40]. By a result of Choi [9, Theorem 3.4], we have an exact sequence of complex vector spaces

$$0 \to T_u U \xrightarrow{Dy} T_z\mathbb{C}_\times^N \xrightarrow{Dg} T_1\mathbb{C}_\times^N \xrightarrow{Dp} \mathbb{C}^k \to 0, \tag{I-2}$$

where p is a certain analytic map defined by Choi in terms of the incidences of the edges of the triangulation to the toroidal ends of M.

On the other hand, the triangulation $\mathcal{T}$ determines a subspace $M_0 \subset M$ obtained by removing disjoint open neighbourhoods of all edges of $\mathcal{T}$ from M. Denote by $\mathscr{K}$ the sheaf of germs of Killing vector fields on M. Since $M_0 \subset M$ is closed, we have a short exact sequence of sheaves $0 \to \mathscr{K}_{M,M_0} \to \mathscr{K} \to \mathscr{K}_{M_0} \to 0$ whose associated cohomology long exact sequence reduces to

$$0 \to H^1(M;\mathscr{K}) \to H^1(M;\mathscr{K}_{M_0}) \to H^2(M;\mathscr{K}_{M,M_0}) \to H^2(M;\mathscr{K}) \to 0. \tag{I-3}$$

The unique finite-volume complete hyperbolic structure on M can always be deformed around each cusp. Algebraically, these deformations give rise to a smooth Euclidean-open neighbourhood of the conjugacy class of the discrete faithful representation in the $PSL_2\mathbb{C}$-character variety of M. This neighbourhood is parametrized, generically in a 2^k:1 manner, by the log-parameters (complex lenghts) in U and contains an open, dense subset corresponding to hyperbolic structures which are incomplete at all ends. With the usual conventions, these structures correspond to log-parameters $u = (u_1, \ldots, u_k)$ satisfying $u_l \neq 0$ for all l.

The following theorem provides a cohomological interpretation of Thurston's gluing equations.

Theorem 1 (Theorem 4.3.1)**.** *Consider on M a hyperbolic structure incomplete at all ends, obtained as a small deformation of the unique complete structure and let u be the corresponding vector of non-zero log-parameters. Then there exists a commutative diagram with exact rows and columns:*

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_u U & \xrightarrow{Dy} & T_z \mathbb{C}^N_\times & \xrightarrow{Dg} & T_1 \mathbb{C}^N_\times & \xrightarrow{Dp} & \mathbb{C}^k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow{\scriptstyle \eta_4} & & \\ 0 & \longrightarrow & H^1(M;\mathscr{K}) & \longrightarrow & H^1(M;\mathscr{K}_{M_0}) & \xrightarrow{\partial} & H^2(M;\mathscr{K}_{M,M_0}) & \longrightarrow & H^2(M;\mathscr{K}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \operatorname{Coker}\eta_2 & \xrightarrow{\cong} & \operatorname{Coker}\eta_3 & \longrightarrow & 0 & & \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & 0 & & 0. & & & & \end{array}$$

We also study the completeness equations themselves from the cohomological point of view. By [9, Theorem 4.13], it suffices to consider one homotopically non-trivial oriented simple closed curve near each end of M. Denote by γ a multicurve consisting of arbitrarily chosen such curves, not necessarily the same as the curves used in the definition of the log-parameter u. The holomorphic derivative of the log-parameter of γ is then a complex-linear map $L_\gamma \circ Dy\colon T_u U \to \mathbb{C}^k$. By Theorem 1, the domain and the codomain of $L_\gamma \circ Dy$ are isomorphic to $H^1(M;\mathscr{K})$ and $H^2(M;\mathscr{K})$ respectively, so we may treat $L_\gamma \circ Dy$ as a function $H^1(M;\mathscr{K}) \to H^2(M;\mathscr{K})$. Consider the maps $r^\bullet\colon H^\bullet(M;\mathscr{K}) \to H^\bullet(M;\mathscr{K}_{\partial M})$ induced by the restriction of the sheaf $\mathscr{K}$ to a collection of tori about the incomplete ends of M. Since r^2 is an isomorphism, a careful analysis of the maps in Theorem 1 reveals the following cohomological interpretation of the log-parameters of peripheral curves.

Theorem 2 (Theorem 4.4.3)**.** *Upon identifying the domain and codomain of $L_\gamma \circ Dy$ with the first and second cohomology groups of $\mathscr{K}$ by the way of Theorem 1, we have*

$$r^2 \circ L_\gamma \circ Dy = r^1(\,\cdot\,) \smile [\gamma]^*,$$

where $[\gamma]^* \in H^1(\partial M;\mathbb{C})$ is the Poincaré dual of the homology class of γ.

Note that the cup product occurring above is $\smile\colon H^1(M;\mathscr{K}_{\partial M}) \times H^1(\partial M;\mathbb{C}) \to H^2(M;\mathscr{K}_{\partial M})$.

Analysis of the 1-loop Conjecture

Using Theorems 1 and 2, we are able to mount a direct attack on the 1-loop Conjecture (I-1), which we remind was formulated in [11] only for $k = 1$.

We start by generalizing the original 1-loop Conjecture (I-1) to the case of any non-trivial peripheral curve γ and state it for any one-cusped hyperbolic 3-manifold M with a positively oriented geometric ideal triangulation. Thus M is no longer assumed to be a complement of a knot in S^3. The conjectural expression can moreover be rewritten more symmetrically as the quotient of two terms,

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) \overset{?}{=} \pm \frac{\frac{1}{2}\det\left[\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right]}{\zeta^{f}\,\zeta'^{f'}\,\zeta''^{f''},} \tag{I-4}$$

in which the symbols ζ, ζ', ζ'' are certain rational functions of the shape parameters[3] and the matrices $\widehat{G}$, $\widehat{G}'$, $\widehat{G}''$ are closely related to the matrices $\mathbf{A}, \mathbf{B}$. The numerator thus corresponds to the main term of the original conjectural formula (I-1), whereas the denominator is the symmetrized 'monomial correction term'.

Our normalization results concerning the non-abelian Reidemeister torsion include in particular the important Corollary 4.4.4 of Theorem 2, guaranteeing that Choi's sequence (I-3) reproduces Porti's choice of bases for the cohomology groups of $\mathscr{K}$ whenever the hyperbolic structure is incomplete at all ends. Moreover, Theorem 1, in combination with a result of Milnor (Theorem 3.1.5) on multiplicativity of torsion with respect to short exact sequences, allows us to decompose the adjoint non-abelian torsion into a product of four terms, which we shall write simply as

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = \mathbb{T}_1\,\mathbb{T}_2\,\mathbb{T}_3\,\mathbb{T}_4, \tag{I-5}$$

while referring to Section 5.2 for details. It is easily proved that $\mathbb{T}_1 = \mathbb{T}_2 = \pm 1$. A slightly more involved, but still completely elementary computation then shows that $\mathbb{T}_3$ is exactly equal to the numerator of (I-4) when $k = 1$. We also propose a closed formula for $\mathbb{T}_3$ when $k > 1$, which holds under a combinatorial assumption on the ideal triangulation, thus generalizing the 1-loop Conjecture to the case of multiple cusps. The generalized formula looks essentially the same as the numerator of (I-4) except that the factor $\frac{1}{2}$ is replaced with $\frac{1}{2^k}$. Lastly, the fourth term $\mathbb{T}_4$ in the above decomposition corresponds to the subspace $M_0 \subset M$ discussed above.

A priori, the decomposition in (I-5) only holds when the hyperbolic structure on M is incomplete at all ends. However, the right-hand side of the equality is an analytic function on the open subset of the $PSL_2\mathbb{C}$-character variety given by such structures, whereas the left hand-side is analytic in a neighbourhood of the conjugacy class of the discrete faithful representation by the work of Porti [46]. Hence, analytic continuation allows us to extend (I-5) also to the structures with one or more true hyperbolic cusps. In this way, we obtain

Theorem 3 (cf. Theorem 5.6.2)**.** *When M has only one cusp and the nontrivial peripheral curve γ is chosen arbitrarily, the equality*

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = \pm \frac{\frac{1}{2}\det\left[\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right]}{\mathbb{T}_4^{-1}} \tag{I-6}$$

holds both at the complete hyperbolic structure and any small deformation of it.

Hence, we are able to reproduce the main term in the conjectural expression (I-4). As an easy corollary, we prove that the conjectural expression is never equal to zero.

The denominator of (I-6) contains the term $\mathbb{T}_4$ which is equal to the Reidemeister torsion of the subspace $M_0 \subset M$, calculated with respect to certain bases $\underline{c}$, $\underline{h}$, the definition of which is also closely related to the gluing equations. The subspace M_0 is considerably simpler than M, since it has the homotopy type of a graph. Moreover, $\mathbb{T}_4$ does not depend in any way on the curve γ.

[3] The cohomological meaning of the numbers ζ, ζ', ζ'' is explained in Section 4.4.2.

Thus, we reduce the original 1-loop Conjecture to a question about the Reidemeister torsion of the free group $\pi_1(M_0)$, asking whether $\mathbb{T}_4^{-1} \stackrel{?}{=} \pm\zeta^f \zeta'^{f'} \zeta''^{f''}$. We also attempt to generalize the above equality to a larger class of local systems.

We finish by providing some further evidence pointing to the validity of the full 1-loop Conjecture, although a complete proof still eludes us.

Normalization of torsion

To prove Theorem 3, we have to adapt Porti's construction of the non-abelian Reidemeister torsion to the setting of local systems. An essential element of the definition of the adjoint torsion is the normalization of bases of the twisted cochain complexes used to compute it. More precisely, Porti [46] describes a distinguished class of bases, called 'geometric bases' which we interpret in a coordinate-free way as follows.

Suppose X is a connected manifold of dimension at most 3 and of Euler characteristic zero. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For a unimodular representation $\varrho\colon \pi_1(X) \to SL(n, \mathbb{F})$, denote by $E = E_\varrho$ the corresponding flat $\mathbb{F}$-vector bundle of rank n. Since $\bigwedge^n E \cong X \times \mathbb{F}$ as a flat bundle, any choice of this isomorphism allows one to identify elements of $\mathbb{F}$ with global, constant (parallel) sections of $\bigwedge^n E$. In what follows, we consider this isomorphism to be fixed once and for all.

Assume now that X is equipped with a preferred cell decomposition. The cellular cochain groups $C^\bullet(X; E)$ can be endowed with $\mathbb{F}$-bases as follows: for every oriented cell s we form a collection of cochains $\{c_s^{(1)}, \dots, c_s^{(n)}\}$ which map s to germs $\{f_s^{(1)}, \dots, f_s^{(n)}\} \subset E_x$ of local, parallel sections of E at an arbitrarily chosen point $x \in s$ and which vanish on all other cells. We further require that $\bigwedge_{i=1}^n f_s^{(i)} = \pm 1$ under the isomorphism $\bigwedge^n E \cong X \times \mathbb{F}$, which in particular implies the linear independence of the c_s's. A geometric basis of $C^\bullet(X; E)$ is then the union

$$\bigcup_{s \text{ a cell of } X} \{c_s^{(1)}, \dots, c_s^{(n)}\} \subset C^\bullet(X; E).$$

As observed by Porti [46], our assumptions on X then imply that the algebraic torsion of $C^\bullet(X; E)$ does not depend on the choice of a geometric basis.

The above construction works equally well when we consider instead the Čech cochain complex $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$, where $\mathfrak{U}$ is a good open cover of X and $\mathscr{F}$ is the sheaf of parallel sections of E. This result is stated as Theorem 3.6.10.

The main application of the above abstract setup is when $X = M$ is an oriented 3-dimensional hyperbolic manifold of finite volume. We set $\mathbb{F} = \mathbb{C}$ and take ϱ to be the 3-dimensional complex representation $\mathrm{Ad} \circ \mathrm{hol}\colon \pi_1(M) \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C})$, where hol is the holonomy representation of the hyperbolic structure. In this case, the sheaf $\mathscr{F} \cong \mathscr{K}$ is the sheaf of germs of Killing vector fields on M [34, Theorem 8.1]. Choosing a fixed basis $\underline{b}$ for the 3-dimensional vector space $\mathfrak{sl}_2\mathbb{C} \cong \mathscr{K}(\mathbb{H}^3)$, we find that any basis of $\check{C}^\bullet(\mathfrak{U}, \mathscr{K})$ consisting locally of pullbacks of the elements of $\underline{b}$ under local orientation-preserving geometric coordinate charts satisfies the above normalization condition. In other words, we can produce 'geometric bases' by purely geometric means.

Chapter 1

Geometric preliminaries

This chapter is a selection of preliminary material concerning certain basic properties of hyperbolic 3-manifolds, ideal triangulations, and Killing vector fields on geometric manifolds. Most of the contents of this chapter are well known and can be found in numerous sources, including in particular the bibliography references [4, 29, 47, 51].

Throughout this thesis, the following standard notations will be used.

- $\mathbb{H}^3$ always stands for the hyperbolic 3-space, equipped with the following orientation. Using the upper-halfspace model, we can write $\mathbb{H}^3 \cong \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$. The orientation is then $1 \wedge i \wedge \frac{\partial}{\partial t}$.
- For a Riemannian manifold M, $\mathrm{Isom}(M)$ will denote the group of isometries of M. When M is oriented, $\mathrm{Isom}^+(M)$ will denote the subgroup of $\mathrm{Isom}(M)$ consisting of orientation-preserving isometries.
- The identification $\mathrm{Isom}^+(\mathbb{H}^3) = PSL_2\mathbb{C}$, which comes from the action of $PSL_2\mathbb{C}$ by analytic automorphisms on the Riemann sphere $\mathbb{C}P^1 = \partial_\infty \mathbb{H}^3$, will be understood. We shall usually write elements of $PSL_2\mathbb{C}$ as Möbius transformations, but sometimes we shall consider them instead as $SL_2\mathbb{C}$ matrices defined only up to sign.

1.1 Killing vector fields

In this section, we review certain basic results concerning *Killing vector fields*, named after the German mathematician Wilhelm Killing (1847–1923). Most of the results discussed here can be found in the research articles of K. Nomizu [43] and Matsushima–Murakami [34].

Let M be a smooth Riemannian manifold with metric g and denote by $\mathscr{X}(M)$ the vector space of all smooth vector fields on M. Roughly speaking, a Killing field is a vector field $X \in \mathscr{X}(M)$ that arises as the velocity field of a one-parameter family of local isometries of M; therefore, it corresponds to an infinitesimal generator of a rigid motion of M.

Since Killing vector fields are a useful tool in the description of deformations of geometric structures [26], in Section 1.1.1 we pay special attention to the geometric properties of Killing vector fields on the hyperbolic 3-space $\mathbb{H}^3$ which serves as a local model for any hyperbolic 3-manifold.

Definition 1.1.1 (Killing Vector Fields)**.** Let M be a smooth manifold with a Riemann metric g. A vector field $X \in \mathscr{X}(M)$ is called a *Killing vector field* if for any $Y, Z \in \mathscr{X}(M)$ we have

$$X(g(Y,Z)) = g\left([X,Y], Z\right) + g\left(Y, [X,Z]\right). \tag{1.1.1}$$

The set of all Killing vector fields on M is denoted by $\mathscr{K}(M)$.

The above definition is taken from [43], where many fundamental properties of Killing fields were first established. We note that in Lemma 2 of [43], it is shown that the vector space $\mathscr{K}(M)$ is a Lie subalgebra of $\mathscr{X}(M)$: if X and Y are Killing fields, then so is $[X,Y]$.

We now wish to highlight the geometric meaning of Definition 1.1.1. Suppose $X \in \mathscr{K}(M)$ and a point $p \in M$ is fixed arbitrarily. Denote by φ the flow of X near p, i.e., a one-parameter family of diffeomorphisms

$$\varphi_t \colon U \to M, \quad t \in (-\varepsilon, \varepsilon)$$

satisfying $\frac{d}{dt}\Big|_{t=0} \varphi_t = X$ on U, where U is an open neighbourhood of p and $\varepsilon > 0$. We can write

$$\begin{aligned} X(g(Y,Z)) &= \frac{d}{dt}\Big|_{t=0} \left[\varphi_t^* \left(g(Y,Z)\right)\right] \\ &= (\mathcal{L}_X g)(Y,Z) + g\left(\frac{d}{dt}\Big|_{t=0} \varphi_{-t_*} Y,\, Z\right) + g\left(Y,\, \frac{d}{dt}\Big|_{t=0} \varphi_{-t_*} Z\right) \\ &= (\mathcal{L}_X g)(Y,Z) + g\left([X,Y], Z\right) + g\left(Y, [X,Z]\right), \end{aligned}$$

where $\mathcal{L}_X g$ denotes the Lie derivative of g along X. Since X satisfies (1.1.1) for any Y and Z, we must have $\mathcal{L}_X g \equiv 0$. In other words, flows of a Killing field preserve the Riemann metric g.

The above observation leads to the interpretation of Killing vector fields as *infinitesimal isometries*. Denote by $\mathrm{Isom}(M)$ the group of all isometries of M, i.e., the group of self-diffeomorphisms of M preserving the Riemann metric. As observed in [43], if the metric g is complete, the Lie algebra $\mathscr{K}(M)$ is isomorphic to the Lie algebra of $\mathrm{Isom}(M)$. Therefore, the elements of $\mathscr{K}(M)$ can be understood as infinitesimal generators of rigid motions of M.

In general, the Lie algebra $\mathfrak{g}$ of a Lie group G depends only on the connected component of identity in G, which we shall denote G_0. When M is oriented, the connected component of identity $\mathrm{Isom}_0(M)$ consists entirely of orientation-preserving isometries: $\mathrm{Isom}_0(M) \subseteq \mathrm{Isom}^+(M)$. Therefore, the Lie algebra of $\mathrm{Isom}(M)$ is isomorphic to that of $\mathrm{Isom}^+(M)$. Assume that M is complete and oriented. Nomizu [43] shows that the identification

$$\mathscr{K}(M) \cong T_{\mathrm{Id}} \, \mathrm{Isom}^+(M) \tag{1.1.2}$$

is, in general, an isomorphism of real Lie algebras.

It is often useful to study locally defined Killing vector fields. If $U \subseteq M$ is open, then U is itself a manifold with the Riemann metric $g\big|_U$. We can therefore consider Killing vector fields defined only on U.

Definition 1.1.2. Let M be a Riemannian manifold. The sheaf $\mathscr{K}$ which assigns to every open set $U \subseteq M$ the space $\mathscr{K}(U)$ of Killing vector fields on U is called the *sheaf of germs of Killing vector fields on M*.

Definition 1.1.3 (Local system)**.** Let $\mathscr{F}$ be a sheaf of abelian groups on a paracompact topological space X. We call $\mathscr{F}$ a *local system* or a *locally constant sheaf* if every point $x \in X$ has an open neighbourhood $U_x \subseteq X$ such that the restriction $\mathscr{F}|_{U_x}$ is isomorphic to a constant sheaf on U_x.

In the following, we shall see that under certain assumptions on the Riemannian manifold M, the sheaf $\mathscr{K}$ is a local system on M. To this end, we need a local analogue of the isomorphism (1.1.2). Recall that the assumption needed for (1.1.2) to hold is that M be complete. However, even if this assumption is not true globally, the local structure of $\mathscr{K}$ may still reflect that of the Lie algebra of some group of local isometries. This problem was studied by K. Nomizu in [43], where he proves that if the manifold M is connected and real-analytic, the dimension of the stalk $\mathscr{K}_x$ at $x \in M$ is constant as a function of x (Theorem 2 in [43]). In fact, Nomizu shows that more is true: if M is real-analytic, every point $x \in M$ has an open neighbourhood U_x such that $\mathscr{K}(U_x) \cong \mathscr{K}_x$.

Obviously, the map $\mathscr{K}(U_x) \to \mathscr{K}_x$ always exists; the non-trivial part of Nomizu's proof is to show that every element $X_x \in \mathscr{K}_x$, which can be thought of as a germ of a Killing field at x, can be extended to a Killing field on an open neighbourhood U_x, yielding the inverse map $\mathscr{K}_x \to \mathscr{K}(U_x)$. Nomizu's approach is to interpret this existence problem as the problem of the existence and uniqueness of solutions to a certain differential equation on M. If $\alpha\colon [0,1] \to M$ is a smooth curve, [43, Lemma 7] implies that a germ of a Killing field $X_0 \in \mathscr{K}_{\alpha(0)}$ can be continued along α. This continuation is unique and invariant under smooth homotopy of α. In particular, if U_x is a connected, simply-connected neighbourhood of a point $x \in M$, every germ $X_x \in \mathscr{K}_x$ can be extended in a unique way to a Killing field $X \in \mathscr{K}(U_x)$.

Matsushima and Murakami [34] add to Nomizu's results by also exploring the algebraic significance of the analytic continuation of Killing fields. In particular, the work of Matsushima and Murakami implies the following theorem.

Theorem 1.1.4 (Matsushima-Murakami). *Let M be a connected hyperbolic 3-manifold. Then*

(i) the sheaf $\mathscr{K}$ is a local system on M.

(ii) $\mathscr{K}$ is isomorphic to the local system defined by the representation $\mathrm{Ad} \circ \mathrm{hol}$, where

$$\mathrm{hol}\colon \pi_1(M) \to PSL_2\mathbb{C}$$

is a holonomy representation of the hyperbolic structure.

The above is essentially Theorem 8.1 in [34], except that we stated it only in the special case of the Riemann symmetric space $\mathbb{H}^3$ and the Lie group $PSL_2\mathbb{C}$. The holonomy representation hol occurring in Part (ii) is defined and discussed in detail in Chapter 2. In particular, in Section 2.3.3 we elaborate on the geometric meaning of the second part of the above theorem.

1.1.1 Geometric interpretation of Killing vector fields on $\mathbb{H}^3$

The goal of this section is to describe the geometric meaning of Killing fields as *infinitesimal isometries* in the specific case of the hyperbolic 3-space $\mathbb{H}^3$. Since $\mathbb{H}^3$ is a complete Riemannian manifold, (1.1.2) holds for $\mathbb{H}^3$.

One unusual feature of this situation is that $\mathrm{Isom}^+(\mathbb{H}^3)$ happens to be a *complex* Lie group, with the complex structure coming from its identification with $PSL_2\mathbb{C}$. Hence, the identification

$$\psi\colon\ \mathfrak{sl}_2\mathbb{C} \xrightarrow{\cong} \mathscr{K}(\mathbb{H}^3), \tag{1.1.3}$$

induces a complex structure on the (*a priori* real) vector space $\mathscr{K}(\mathbb{H}^3)$ of Killing vector fields on $\mathbb{H}^3$. We are going to describe the isomorphism ψ in more detail below.

The Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is the three-dimensional complex Lie algebra of traceless 2×2 matrices. It is spanned by the following basis elements

$$\mathfrak{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{h} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad \mathfrak{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.1.4}$$

To better understand the isomorphism ψ of (1.1.3), we are going to determine the one-parameter subgroups of $PSL_2\mathbb{C}$ generated by the basis elements $\mathfrak{e}$, $\mathfrak{h}$, $\mathfrak{f}$.

Unlike most texts on differential geometry, we wish to work over $\mathbb{C}$, since our goal is to make use of the analytic structure of $PSL_2\mathbb{C}$. Thus, the subgroups of $PSL_2\mathbb{C}$ which we are going to describe are in fact one-*complex*-parameter subgroups. Since it is easier to work with matrices, we shall first find the one-parameter subgroups of $SL_2\mathbb{C}$ generated by $\mathfrak{e}$, $\mathfrak{h}$, $\mathfrak{f}$ using the matrix exponential $\exp\colon \mathfrak{sl}_2\mathbb{C} \to SL_2\mathbb{C}$, and then take their images under the quotient map $q\colon SL_2\mathbb{C} \to PSL_2\mathbb{C}$.

To start with, for any $t \in \mathbb{C}$, we have

$$\exp(t\mathfrak{e}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where we have used the nilpotence property $\mathfrak{e}^2 = 0$ to truncate the power series after the first-order term. Similarly, using $\mathfrak{f}^2 = 0$, we find

$$\exp(t\mathfrak{f}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Finally,

$$\begin{aligned} \exp(t\mathfrak{h}) &= \sum_{n=0}^{\infty} \frac{1}{n!}(t\mathfrak{h})^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{t}{2} & 0 \\ 0 & \frac{-t}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (\frac{t}{2})^2 & 0 \\ 0 & (\frac{-t}{2})^2 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}. \end{aligned} \tag{1.1.5}$$

Applying the quotient map q to the matrices found above, we obtain the following elements of $PSL_2\mathbb{C}$:

$$\pm\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : z \mapsto z + t, \quad \pm\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : z \mapsto \frac{z}{tz+1}, \quad \pm\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} : z \mapsto e^t z.$$

The action of the above Möbius transformations on $\mathbb{C}P^1 = \partial_\infty\mathbb{H}^3$ can be described geometrically as follows.

- $z \mapsto z + t$ acts as a translation in the plane $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$. The amount of translation is simply the complex number t.

- When $t \in i\mathbb{R}$ is purely imaginary, $z \mapsto e^t z$ acts as a rotation about 0 through the angle $\operatorname{Im} t$. For $t \in \mathbb{R}$, $z \mapsto e^t z$ is a homothety with a stretching factor of e^t. Note that in both cases

$z \mapsto e^t z$ preserves the points 0 and ∞ in $\mathbb{C}P^1$. For a general $t = x + iy$, $z \mapsto e^t z = e^x(e^{iy}z)$ is a composition of a rotation about 0 with a homothety.

- $z \mapsto \frac{z}{tz+1}$ is harder to describe directly, but simplifies when conjugated with the complex inversion $z \mapsto \frac{1}{z}$, giving the Möbius transformation

$$z \mapsto \frac{t\frac{1}{z} + 1}{\frac{1}{z}} = z + t,$$

which is just a translation of $\mathbb{C}$. Since the involution $z \mapsto \frac{1}{z}$ exchanges 0 and ∞, we can interpret $z \mapsto \frac{z}{tz+1}$ as a translation in the 'antipodal' copy of $\mathbb{C}$, embedded as $\mathbb{C}P^1 \setminus \{0\}$.

In order to understand the map ψ of (1.1.3) geometrically, we use the identity

$$v = \frac{d}{dt}\Big|_{t=0} (q \circ \exp(tv)), \quad v \in \mathfrak{sl}_2\mathbb{C},\ t \in \mathbb{C} \tag{1.1.6}$$

in which $\exp\colon \mathfrak{sl}_2\mathbb{C} \to SL_2\mathbb{C}$ is the matrix exponential and $q \circ \exp$ is the actual Lie group exponential map $\mathfrak{sl}_2\mathbb{C} \to PSL_2\mathbb{C}$ which is of interest to us. For any $t \in \mathbb{C}$, the Möbius transformation $q \circ \exp(tv)$ acts on $\mathbb{H}^3$ as a hyperbolic isometry. However, since t is complex, the holomorphic derivative (1.1.6) cannot be used directly to define a unique (real) vector field in $\mathscr{X}(\mathbb{H}^3)$. This is a minor difficulty though, as every complex vector space is a real vector space of twice the dimension. Hence, it suffices to consider the derivatives with respect to a real parameter x.

For any $v \in \mathfrak{sl}_2\mathbb{C}$, the Killing vector field $\psi(v) \in \mathscr{K}(\mathbb{H}^3)$ is given at a point $p \in \mathbb{H}^3$ by

$$\psi(v)_p = \frac{d}{dx}\Big|_{x=0} (q \circ \exp(xv)) \bullet p, \quad v \in \mathfrak{sl}_2\mathbb{C},\ x \in \mathbb{R}, \tag{1.1.7}$$

where the $\bullet$ stands for the action of $PSL_2\mathbb{C}$ on $\mathbb{H}^3$ by hyperbolic isometries.

We turn to a description of geometric properties of the images under ψ of the basis vectors (1.1.4) and their complex multiples. We start with the one-dimensional subspace $\mathbb{C}\psi(\mathfrak{e}) \subset \mathscr{K}(\mathbb{H}^3)$. In the upper halfspace model of $\mathbb{H}^3$, the boundary plane is a copy of $\mathbb{C}$ on which the Möbius transformations $z \mapsto z + t$ act as translations. These translations extend to horizontal translations of the upper-halfspace model. Therefore, we interpret $\mathbb{C}\psi(\mathfrak{e})$ as the complex vector space of *infinitesimal horizontal translations* of the upper-halfspace model. Since these horizontal translations preserve all horospheres centered at infinity, the same must be true in the Poincaré ball model, as shown in Figure 1.1.1, top.

The interpretation of the complex subspace $\mathbb{C}\psi(\mathfrak{f}) \subset \mathscr{K}(\mathbb{H}^3)$ is similar, except that these infinitesimal translations preserve the horospheres centered at zero. The upper-halfspace model's lack of symmetry makes it somewhat harder to visualise these infinitesimal isometries, so we prefer to use the Poincaré ball model. The bottom part of Figure 1.1.1 shows one of the horospheres centered at zero and illustrates that $\psi(\mathfrak{f})$ becomes an ordinary infinitesimal translation when that horosphere is isometrically identified with $\mathbb{C}$.

Finally, we consider the $\mathbb{C}$-span of $\psi(\mathfrak{h}) \in \mathscr{K}(\mathbb{H}^3)$. We have observed above that the one-complex-parameter family of Möbius transformations generated by $\mathfrak{h}$ fixes $\{0, \infty\} \subset \mathbb{C}P^1$, whence the corresponding isometries of $\mathbb{H}^3$ must preserve the geodesic $(0, \infty)$.

Consider again the upper halfspace model of $\mathbb{H}^3$. For t real, $t\mathfrak{h}$ exponentiates to the homothety $z \mapsto e^t z$ of the horizontal boundary plane, which extends to $\mathbb{H}^3$ as a translation along the geodesic $(0, \infty)$. The translation distance is $|t|$; when $t > 0$, the translation is upwards, and when $t < 0$, it

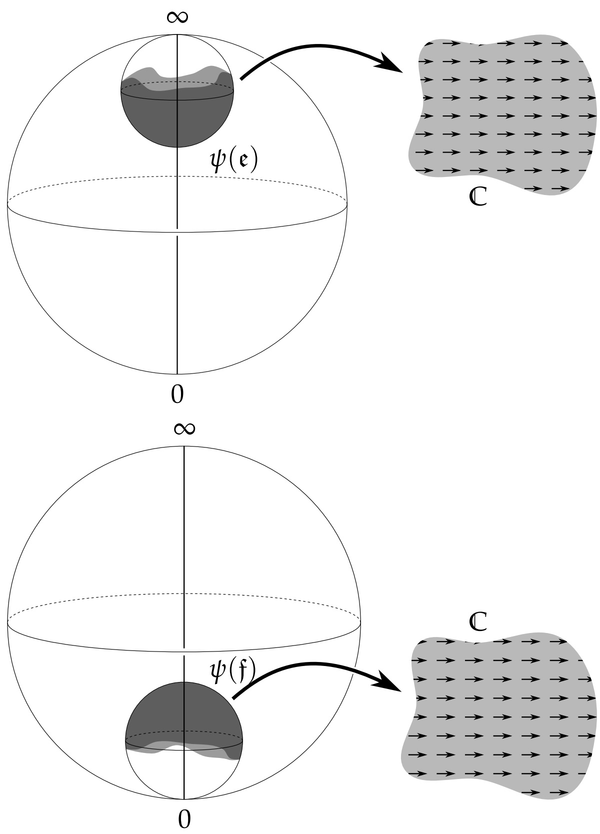


Figure 1.1.1: TOP: The Killing vector field $\psi(\mathfrak{e})$ restricted to a horosphere centered at infinity. When a portion of the horosphere is mapped isometrically to $\mathbb{C}$, the push-forward of $\psi(\mathfrak{e})$ becomes an infinitesimal translation of $\mathbb{C}$ (hence a constant holomorphic vector field on $\mathbb{C}$). BOTTOM: The Killing vector field $\psi(\mathfrak{f})$ on a horosphere at zero. The Möbius transformation $z \mapsto 1/z$ exchanges horospheres at zero and infinity, taking $\psi(\mathfrak{e})$ to $\psi(\mathfrak{f})$ and vice versa.

is downwards. Therefore, $\psi(\mathfrak{h})$ can be interpreted as a unit speed *infinitesimal translation* along the geodesic $(0, \infty)$, towards infinity. Figure 1.1.2 illustrates the resulting vector field in the ball model. If, on the other hand, t is purely imaginary, then $\psi(t\mathfrak{h})$ is an *infinitesimal rotation* about the geodesic $(0, \infty)$. Figure 1.1.3 illustrates the field $\psi(i\mathfrak{h})$ in the equatorial plane of the Poincaré ball.

For a general $t \in \mathbb{C}$, the vector field $\psi(t\mathfrak{h})$ can therefore be interpreted as an *infinitesimal corkscrew motion* about the geodesic $(0, \infty)$, with the real and imaginary parts of t determining the translational and rotational components of $\psi(t\mathfrak{h})$, respectively.

1.1.2 Local structure of Killing fields in the neighbourhood of a line

In the preceding section, we paid special attention to the local properties of Killing fields in a neighbourhood of the infinite geodesic $(0, \infty) \subset \mathbb{H}^3$. We wish to generalize these geometric observations to an arbitrary simple infinite geodesic L in an oriented hyperbolic 3-manifold M.

Using the upper halfspace model of $\mathbb{H}^3$, we would like to build a local geometric chart φ on M which maps L to the infinite geodesic $(0, \infty) := \{(0, t) \mid t > 0\} \subset \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3$.

Since M is locally simply connected, there exists an open neighbourhood U_L of L and an orientation-preserving geometric chart

$$\varphi \colon U_L \to \mathbb{H}^3$$

satisfying $\varphi(L) = (0, \infty)$. However, φ is far from unique. To start with, if L is oriented, φ might take that orientation to either of the two possible orientations of $(0, \infty)$. We therefore fix an

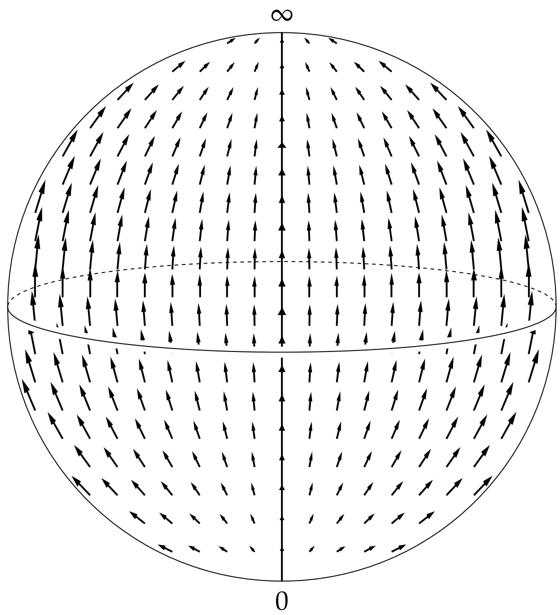


Figure 1.1.2: The Killing vector field $\psi(\mathfrak{h})$ is an infinitesimal unit speed translation along the geodesic $(0,\infty)$, shown here in the Poincaré ball model. The vector field depicted is the restriction of $\psi(\mathfrak{h})$ to the plane of the figure. Since the field has rotational symmetry, we would see the same picture in every hyperbolic plane in $\mathbb{H}^3$ containing the line $(0,\infty)$.

orientation of L and require φ to take it to the positive orientation of $(0,\infty)$, i.e., the orientation from 0 to ∞. Even so, φ is still not unique, since if $\alpha \in PSL_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3)$ is any isometry of $\mathbb{H}^3$ mapping the geodesic $(0,\infty)$ to itself while preserving its orientation, $\alpha \circ \varphi$ is another local chart satisfying our criteria. The most general such isometry is the Möbius transformation

$$\alpha\colon z \mapsto \lambda z, \quad \text{where } \lambda \in \mathbb{C} \setminus \{0\}. \tag{1.1.8}$$

Conversely, suppose φ and φ' are two orientation-preserving geometric charts which take L to $(0,\infty)$ with the preferred orientations. Then $\varphi' \circ \varphi^{-1}$ is a locally defined orientation-preserving isometry which leaves $(0,\infty)$ along with its orientation invariant, and hence extends to a unique isometry of $\mathbb{H}^3$ of the form (1.1.8). In other words, $\varphi' = \alpha \circ \varphi$ for some α. We have shown that any two oriented parametrizations of L are related by an isometry α of the form (1.1.8) for some $\lambda \in \mathbb{C} \setminus \{0\}$.

We now study the effect of this indeterminacy of the chart φ on Killing fields. Given a Killing vector field $X \in \mathscr{K}(\mathbb{H}^3)$, we can push it forward by φ^{-1} to obtain a local Killing field $(\varphi^{-1})_* X$ on $U_L \subset M$. If we change the parametrization, i.e., replace φ with $\varphi' = \alpha \circ \varphi$, we obtain

$$(\varphi')^{-1}_* X = (\alpha \circ \varphi)^{-1}_* X = (\varphi^{-1})_* \left(\alpha^{-1}_* X\right) = (\varphi^{-1})_* \left(\psi \,\mathrm{Ad}(\alpha)^{-1} \psi^{-1} X\right),$$

where ψ is the isomorphism in (1.1.3).

Expressing $\mathrm{Ad}(\alpha) \in \mathrm{Aut}\,\mathfrak{sl}_2\mathbb{C}$ as a 3×3 matrix in the basis $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f}) \subset \mathfrak{sl}_2\mathbb{C}$ gives

$$\mathrm{Ad}(\alpha)\Big|_{\text{basis } \mathfrak{e},\mathfrak{h},\mathfrak{f}} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \tag{1.1.9}$$

(cf. also equation (2.3.7), where α is called H_λ). From the form of this matrix, we see that the

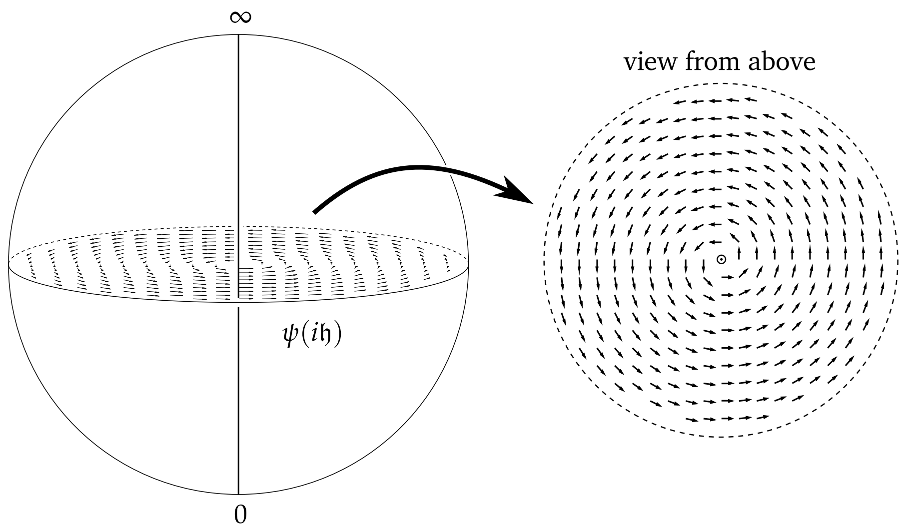


Figure 1.1.3: The Killing vector field $\psi(i\mathfrak{h})$ is an infinitesimal rotation about the geodesic $(0,\infty)$, shown here in the Poincaré ball model. On the right side, we see a hyperbolic plane orthogonal to the line $(0,\infty)$, with the intersection point mapped to the centre of the disc.

vector $\mathfrak{h}$ is fixed by $\mathrm{Ad}(\alpha)$, regardless of the value of λ. In other words,

$$(\varphi'^{-1})_*\psi(\mathfrak{h}) = (\varphi^{-1})_*\psi(\mathfrak{h}). \tag{1.1.10}$$

Treating $\mathfrak{h}$ as an element of $\mathscr{K}(\mathbb{H}^3)$ under the isomorphism ψ, the complex span $\mathbb{C}\psi(\mathfrak{h}) \subset \mathscr{K}(\mathbb{H}^3)$ consists of infinitesimal corkscrew motions about the geodesic $(0,\infty)$. Therefore, the push-forwards of these Killing fields by φ^{-1} are infinitesimal corkscrew motions about the geodesic $L \subset M$. Moreover, (1.1.10) says that this push-forward does not depend on the choice of φ.

The effect of reversing the orientation of L on this construction is easily observed. After a composition with a Möbius transformation α of the form (1.1.8), every hyperbolic isometry preserving the geodesic $(0,\infty) \subset \mathbb{H}^3$ but reversing its orientation can be turned into the Möbius transformation $z \mapsto 1/z$ whose adjoint action on $\mathfrak{sl}_2\mathbb{C}$ takes $\mathfrak{h}$ to $-\mathfrak{h}$. Hence, changing the orientation of L replaces the field $(\varphi^{-1})_*\psi(\mathfrak{h})$ with its negative. This is exactly what we expected from the the geometric interpretation of the push-forward field as an infinitesimal translation in the direction of the orientation of L.

We summarize these elementary observations in the following proposition.

Proposition 1.1.5. *Let M be an oriented hyperbolic 3-manifold and L an infinite simple oriented geodesic in M. Given a sufficiently small open neighbourhood U_L of L in M, there is a well-defined Killing vector field $\mathfrak{h}_L \in \mathscr{K}(U_L)$ given by*

$$\mathfrak{h}_L = (\varphi^{-1})_*\psi(\mathfrak{h})$$

for any orientation-preserving geometric chart $\varphi\colon U_L \to \mathbb{H}^3$ which takes L to $(0,\infty)$ with the upward orientation. $\mathfrak{h}_L$ is a unit speed infinitesimal translation in the direction of the orientation of L and the complex span of $\mathfrak{h}_L$ consists of infinitesimal corkscrew motions about L.

Remark 1.1.6. We remark that Proposition 1.1.5 could be stated more generally for a simple geodesic arc $\gamma\colon (0,1) \to M$ and its isometric identification with an open sub-arc of $(0,\infty) \subset \mathbb{H}^3$.

Moreover, instead of formulating the proposition in terms of open neighbourhoods, we could have stated it in terms of germs of Killing vector fields, i.e., the stalks of the sheaf $\mathscr{K}$ at points of L. Nomizu's study of analytic continuations of Killing vector fields [43] implies that every such germ can be uniquely extended onto a simply-connected neighbourhood of L.

Remark 1.1.7. In [26], Craig Hodgson and S. Kerckhoff discuss a somewhat similar construction: given a point p in a hyperbolic 3-manifold M, they decompose the stalk $\mathscr{K}_p$ as $T_pM \oplus A_p$, interpreting a tangent vector $v_p \in T_pM$ as an infinitesimal translation of M with velocity v_p and defining the complementary space A_p as the space of infinitesimal rotations about p. This approach is also present in the work of Nomizu [43].

Hodgson and Kerckhoff also observe that a complex structure on $\mathscr{K}_p$ can be constructed by sending a tangent vector $v_p \in T_pM$ to an infinitesimal rotation with v_p as an axis, effectively interpreting $\mathscr{K}_p$ as a complexification of T_pM with 'imaginary part' A_p.

Algebraically, the stabilizer of a point $\widetilde{p} \in \mathbb{H}^3$ is a subgroup of $PSL_2\mathbb{C}$ isomorphic to $SO(3)$. Treating $\mathbb{H}^3$ as the coset space $PSL_2\mathbb{C}/SO(3)$ and differentiating the quotient map at the identity, we obtain the short exact sequence of *real* vector spaces

$$0 \to \mathfrak{so}(3) \to \mathfrak{sl}_2\mathbb{C} \to T_{\widetilde{p}}\mathbb{H}^3 \to 0, \tag{1.1.11}$$

with $A_{\widetilde{p}} \cong \mathfrak{so}(3)$. When M is complete of finite volume, we have $M = \mathbb{H}^3/\Gamma$ for a torsion-free Kleinian group Γ. Under this quotient map, the splitting $\mathscr{K}_{\widetilde{p}} = T_{\widetilde{p}}\mathbb{H}^3 \oplus A_{\widetilde{p}}$ descends to the decomposition $\mathscr{K}_p = T_pM \oplus A_p$, where p is the image of $\widetilde{p}$ in M.

Our construction is more limited in scope, since we only consider points on a chosen geodesic L in M. In each tangent space T_pM for $p \in L$, we consider only the one-dimensional subspace $T_pL \subset T_pM$, identifying the complex span of $\mathfrak{h}_L$ as infinitesimal corkscrew motions about L. Note that we do not introduce any new complex structure on the sheaf $\mathscr{K}$, but merely describe the complex structure induced by (1.1.2).

1.2 Ideal triangulations and gluing equations

In this section, we discuss a method of finding hyperbolic structures on 3-manifolds triangulated into ideal tetrahedra. This construction is due to W. Thurston [50]. In Thurston's method, hyperbolicity conditions are expressed with the help of *gluing equations*, the properties of which shall be of particular interest to us. Our discussion of gluing equations is based on the formulation by W. Neumann and D. Zagier [40]. The entire content of this section is well known to experts and is included here primarily for reference and to establish notations and terminology that will be used subsequently.

Suppose that M is a non-compact 3-manifold homeomorphic to the interior of a compact 3-manifold $\bar{M}$ with $\partial\bar{M}$ a union of k torus components, $k > 0$. Let $\mathcal{T}$ be a topological ideal triangulation of M, i.e., a decomposition of M into N ideal tetrahedra $\{\Delta_j\}_{j=1}^N$ glued along their faces.

Topologically, an ideal tetrahedron is simply a tetrahedron with vertices removed. We can equip an ideal tetrahedron with a combinatorial orientation by numbering its ideal vertices (up to an even permutation). A chosen orientation then induces orientations on the faces of the tetrahedron. The gluing is *oriented* if all face identifications reverse the induced orientations of the faces. In this way, an oriented gluing pattern induces a (PL) orientation on M. In what follows, we shall assume that the triangulation $\mathcal{T}$ has an oriented gluing pattern; this implies in particular that M is orientable.

The essence of Thurston's approach is to use the ideal tetrahedra Δ_j as 'hyperbolic pieces' and to build a hyperbolic structure on M by 'gluing' hyperbolic metrics on the Δ_j's. For this purpose, we wish to realize each oriented tetrahedron Δ_j as a positively oriented ideal tetrahedron in $\mathbb{H}^3$ (see Figure 1.2.1). Up to an orientation-preserving isometry, a hyperbolic ideal tetrahedron is determined by a single complex number, called the *shape parameter*. Gluing equations, featuring the shape parameters as the unknowns, express the condition that the hyperbolic metrics on the ideal tetrahedra match up to produce a non-singular hyperbolic metric on M.

An ideal triangulation $\mathcal{T}$ with N tetrahedra also has N edges [9, Proposition 2.2]. We shall write $\{e_i\}_{i=1}^N$ for the set of edges. Note that the choice of the numbering of the tetrahedra and the edges of $\mathcal{T}$ by integers $\{1, \dots, N\}$ is arbitrary; this choice does not affect the arguments that follow.

To formulate Thurston's gluing equations on $\mathcal{T}$, we label pairs of opposite edges of each tetrahedron Δ_j with shape parameters z_j, z_j', z_j'' which are related by

$$z_j' = \frac{1}{1 - z_j}, \quad z_j'' = 1 - \frac{1}{z_j}. \tag{1.2.1}$$

In particular, the three shape parameters associated to every tetrahedron satisfy

$$z_j z_j' z_j'' = -1. \tag{1.2.2}$$

Let $P = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper halfplane in $\mathbb{C}$. The assumption $z_j \in P$ for all j guarantees that the combinatorial orientations of the ideal tetrahedra of $\mathcal{T}$ can be replaced with compatible smooth orientations. To this end, we identify each abstract ideal tetrahedron with a geometric ideal tetrahedron in the hyperbolic 3-space $\mathbb{H}^3$ (see Figure 1.2.1, left). It suffices to consider the ideal tetrahedra with ideal vertices $(0, 1, \infty, s)$ where $s \in P$, requiring the identification to take the combinatorial orientation to the above ordering of ideal vertices, up to an even permutation.

The choice, for every Δ_j, of a pair of opposite edges which are to be labeled z_j, determines the assignment of the labels z_j' and z_j'' by the clockwise rule illustrated in Figure 1.2.1, right. Thus, there are three possible label assignments for each tetrahedron, corresponding to the choice of the initial pair of opposite edges or, equivalently, of a normal quadrilateral type in Δ_j [52, Section 2.2]. We shall henceforth assume that these assignments are fixed in an arbitrary way. Whenever we use the notation $z_j^\square$, it is to be understood that $\square$ is any fixed choice of a normal quadrilateral in Δ_j. Hence, $z_j^\square$ will always be an element of the set $\{z_j, z_j', z_j''\}$.

The precise form of the gluing equations that we shall require is as follows. For each edge e_i of $\mathcal{T}$, we have an *edge consistency equation* of the form

$$e_i \colon \quad \prod_{j=1}^N z_j^{G_{ij}} z_j'^{G_{ij}'} z_j''^{G_{ij}''} = 1, \tag{1.2.3}$$

where $G_{ij}, G_{ij}', G_{ij}'' \in \{0, 1, 2\}$ count the number of times an edge labeled z_j, z_j' or z_j'', respectively,

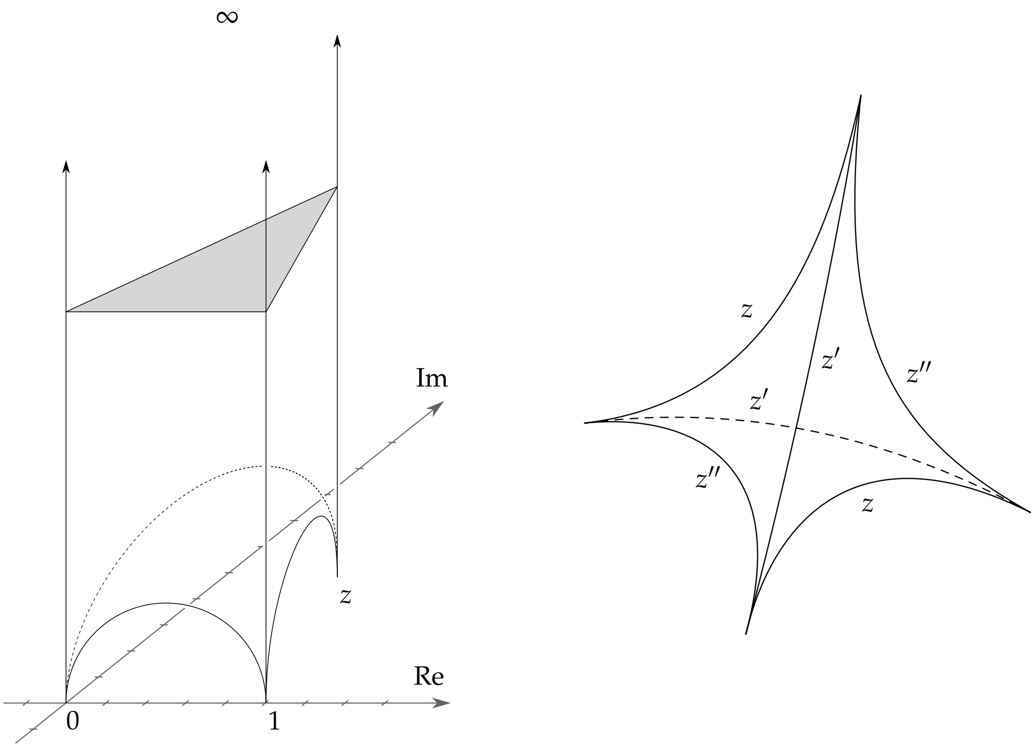


Figure 1.2.1: LEFT: A hyperbolic ideal tetrahedron in the upper-halfspace model of $\mathbb{H}^3$. Every ideal tetrahedron in $\mathbb{H}^3$ can be positioned with three of its vertices at $0, 1, \infty \in \mathbb{C} \cup \{\infty\} = \partial_\infty \mathbb{H}^3$ and the fourth vertex somewhere in the upper halfplane P. The coordinate of this fourth vertex, denoted z, is by definition the shape parameter associated to the edge $(0, \infty)$. The horizontal shaded triangle is similar to the triangle in $\mathbb{C}$ with vertices $(0, 1, z)$. RIGHT: A hyperbolic ideal tetrahedron in the Poincaré ball model of $\mathbb{H}^3$, with a labeling of its edges by shape parameters z, z', z''.

occurs among the edges of the tetrahedra of $\mathcal{T}$ which are glued to become e_i.

We stress that the equations (1.2.3) constitute necessary, but not sufficient conditions for the existence of a non-singular hyperbolic structure on M. Indeed, as seen on the left of Figure 1.2.1, the dihedral angle along an edge labeled z_j is $\operatorname{Arg} z_j$. In order for the angles around each edge of $\mathcal{T}$ to add up to 2π, we must make sure that the equations (1.2.3) hold in their logarithmic form

$$e_i: \quad \sum_{j=1}^{N} G_{ij} \log z_j + G'_{ij} \log z'_j + G''_{ij} \log z''_j = 2\pi\sqrt{-1}, \tag{1.2.4}$$

where log stands for the principal branch of the logarithm on P and the square root of -1 has positive imaginary part. The coefficients in the above equations can be assembled into $N \times N$ matrices $G = \left[G_{ij}\right]$, $G' = \left[G'_{ij}\right]$, $G'' = \left[G''_{ij}\right]$ which only depend on the combinatorics of the ideal triangulation $\mathcal{T}$ and the numbering choices. We shall call G, G', G'' the *gluing matrices*; their properties have been studied in [40] and [41]. Using the gluing matrices and writing

$$Z = (\log z_1, \ldots, \log z_N)^\top, \quad Z' = (\log z'_1, \ldots, \log z'_N)^\top, \quad Z'' = (\log z''_1, \ldots, \log z''_N)^\top, \tag{1.2.5}$$

where log is again the principal branch of the logarithm on P, we arrive at the following concise form of the gluing equations:

$$GZ + G'Z' + G''Z'' = (2\pi i, \ldots, 2\pi i)^\top, \tag{1.2.6}$$

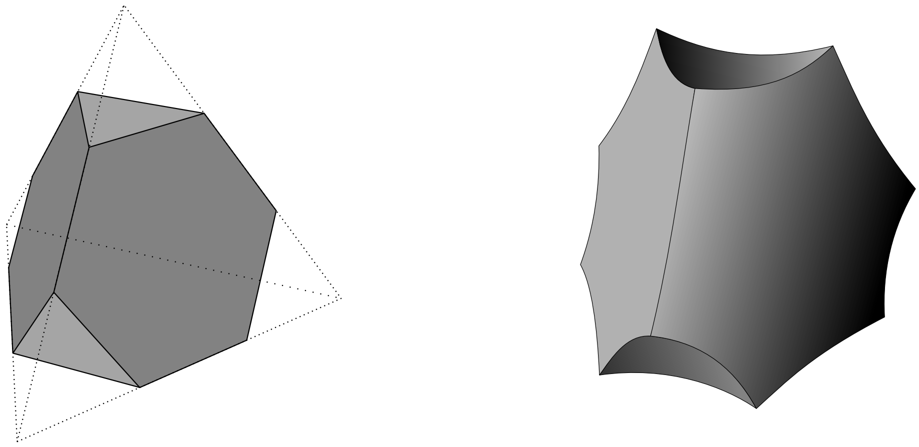


Figure 1.2.2: LEFT: A Euclidean truncated tetrahedron is obtained from an ordinary tetrahedron (dotted) by slicing off the vertices with planes. This process turns triangular faces into hexagons and introduces new faces – the lightly shaded cut-off triangles. RIGHT: A hyperbolic truncated ideal tetrahedron in the Poincaré ball model of $\mathbb{H}^3$. The boundaries of the removed horoball neighbourhoods of the ideal vertices are horospherical triangles inheriting *Euclidean* geometry from their embedding in $\mathbb{H}^3$.

where $i = \sqrt{-1}$ is the imaginary unit.

In order to find the *complete* hyperbolic structure on M, we need to impose additional conditions on the boundary of $\bar{M}$. This boundary can be constructed by removing open neighbourhoods of the ends of M. The compact manifold thus obtained is homeomorphic to $\bar{M}$ and has k boundary components, all of which are tori. This operation turns the tetrahedra of $\mathcal{T}$ into *truncated tetrahedra* (see Figure 1.2.2). The torus components of $\partial\bar{M}$ are triangulated into the cut-off triangles of the truncated tetrahedra.

Completeness of the hyperbolic structure on M requires each boundary component of $\partial\bar{M}$ to become a Euclidean torus. The shapes of the cut-off triangles (i.e., their oriented similarity classes) are given by the shape parameters of the corresponding tetrahedra. One such triangle is visualized in Figure 1.2.1, left.

Assumption 1.2.1. Let $\theta = \{\theta_l\}_{l=1}^k$ be a collection of oriented simple closed curves, one in each component of $\partial\bar{M}$. We assume that θ is *homotopically nontrivial*, which means that $\theta_l \subset T_l$ defines a non-trivial element of $H_1(T_l;\mathbb{C})$ for every l, where T_l is the lth toroidal component of $\partial\bar{M}$.

We may furthermore assume that θ intersects each face of the triangulation $\mathcal{T}$ transversely. Using the method detailed in [40], we can write down a gluing equation for each of the curves θ_l. For this purpose, we define integers $G_{\theta_l,j}$, $G'_{\theta_l,j}$, $G''_{\theta_l,j}$, where $l \in \{1,\dots,k\}$, $j \in \{1,\dots,N\}$, recording the way in which the curve θ_l passes through each triangle of the boundary triangulation. Every time θ_l encircles an angle of a triangle labeled $z_j^\square$, it contributes either $+1$ or -1 to the number $G^\square_{\theta_l,j}$, with the sign depending on the direction of winding (see Figure 1.2.3). Using these combinatorially defined coefficients, we may now formulate the *completeness equations*

$$\theta_l\colon \quad \sum_{j=1}^{N} G_{\theta_l,j}\log z_j + G'_{\theta_l,j}\log z'_j + G''_{\theta_l,j}\log z''_j = 0, \tag{1.2.7}$$

in which we again use the standard branch of the logarithm on P. Note that reversing the orientation of the curve θ_l replaces the coefficients of the above equation with their negatives.

A collection of shape parameters $(z_1,\dots,z_N) \in P^N$ satisfying both (1.2.4) and (1.2.7) de-

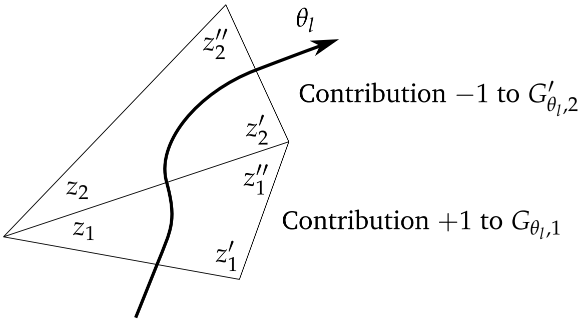


Figure 1.2.3: A curve θ_l passes through two triangles of the boundary torus T_l. This contributes $+1$ to the number $G_{\theta_l,1}$, since the vertex labeled z_1 is passed counter-clockwise, and -1 to the number $G'_{\theta_l,2}$, since the vertex labeled z_2' is passed clockwise. The numbers $G_{\theta_l,j}$, $G'_{\theta_l,j}$, $G''_{\theta_l,j}$ are obtained as sums of all such contributions along θ_l.

termines a complete hyperbolic structure of finite volume on M. This structure can be obtained by realising each tetrahedron Δ_j as a hyperbolic ideal tetrahedron with shape parameter z_j and gluing these tetrahedra along their faces. We refer to [50] for more details.

Remark 1.2.2. We remark that we have used exactly one arbitrarily chosen curve θ_l in each boundary torus. This is possible thanks to a result of Eun-Young Choi [9, Corollary 4.14] which holds under the assumption that all shape parameters z_j have positive imaginary parts. For this reason, we work exclusively with shape parameters in $P = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

Following [9], we are going to assemble the edge constraints ocurring on the left-hand side of (1.2.3) into a single map. This map is defined as

$$g\colon \mathbb{C}_\times^N \to \mathbb{C}_\times^N, \quad g(z_1, \dots, z_N) = \Big(\prod_{j=1}^N z_j^{G_{ij}} z_j'^{G'_{ij}} z_j''^{G''_{ij}}\Big)_{i=1}^N. \tag{1.2.8}$$

As a consequence of (1.2.1), all denominators ocurring in the above expression for g consist only of factors of the form z_j or $(1 - z_j)$. Thus, g is meromorphic on $\mathbb{C}^N$ and holomorphic on P^N.

Definition 1.2.3 (Gluing variety)**.** The *gluing variety* $\mathcal{V}_\mathcal{T}$ associated to the ideal triangluation $\mathcal{T}$ of a 3-manifold M is defined by

$$\mathcal{V}_\mathcal{T} = g^{-1}(1, \dots, 1) \subset \mathbb{C}^N,$$

where N is the number of tetrahedra in $\mathcal{T}$. We also define the *positive part* of the gluing variety,

$$\mathcal{V}_\mathcal{T}^+ = \mathcal{V}_\mathcal{T} \cap P^N.$$

Since the gluing equations (1.2.3) are equivalent to a system of polynomial equations, the use of the term *variety* is justified: $\mathcal{V}_\mathcal{T}$ is an affine algebraic variety over $\mathbb{C}$. In this work, we shall take the analytic point of view and treat $\mathcal{V}_\mathcal{T}$ and $\mathcal{V}_\mathcal{T}^+$ as analytic manifolds. Most of the time, we shall only consider solutions of the gluing equations with positive imaginary parts, effectively restricting the domain of g to just P^N. Note that every point of $\mathcal{V}_\mathcal{T}^+$ gives rise to a hyperbolic structure on M, in general incomplete. Mostow–Prasad Rigidity [4, 29] implies that at most finitely many points of $\mathcal{V}_\mathcal{T}^+$ satisfy the completeness equations (1.2.3). Given a point $z_* \in \mathcal{V}_\mathcal{T}^+$ defining the complete hyperbolic structure on M, the vicinity of z_* in $\mathcal{V}_\mathcal{T}^+$ defines an analytic family of incomplete finite volume hyperbolic structures on M. The study of these deformed structures was initiated by Thurston [50] and continued by Neumann and Zagier [40]; see also Section 1.3 below.

It is not true in general that $\mathcal{V}_{\mathcal{T}}^{+}$ is nonempty or that it contains a solution of the completeness equations, so we would like to single out the triangulations which "detect" the complete hyperbolic structure.

Definition 1.2.4 (Geometric ideal triangulation)**.** Let M be a non-compact 3-manifold homeomorphic to the interior of a compact 3-manifold $\bar{M}$ with $\partial\bar{M}$ a union of k torus components, where $k > 0$. A topological ideal triangulation $\mathcal{T}$ of M with an oriented gluing pattern is called a *geometric triangulation* if the gluing equations (1.2.3) and (1.2.7) on $\mathcal{T}$ have a joint solution $z_* \in P^N$.

Remark 1.2.5. Since the face identifications are orientation-reversing, the positive orientations of the ideal tetrahedra match up to produce an orientation on M. Hence, if M admits a geometric ideal triangulation, then M is orientable. Some authors do not include this requirement of orientability in the definition of a geometric ideal triangulation.

Remark 1.2.6 (The case of $N = 1$)**.** According to B. Burton's census of cusped hyperbolic manifolds [7], the only hyperbolic 3-manifold which can be obtained by gluing the faces of a single ideal tetrahedron is the Gieseking manifold. This manifold is non-orientable; its orientable double cover is the complement of the figure-eight knot 4_1. As a consequence, if $\mathcal{T}$ is a geometric ideal triangulation, then $N \geq 2$.

We finish our discussion of gluing equations on ideal triangulations by providing two examples. The first one of these examples dates back to the early work of Thurston on hyperbolic 3-manifolds, whereas the second one has only been discovered recently.

1.2.1 Example: The standard triangulation of the figure-eight knot complement

We are going to discuss the two-tetrahedron triangulation of the complement of the figure-eight knot 4_1 in the 3-sphere discovered by Thurston [50]. We refer to [5, 29] for detailed descriptions of this construction and proofs that the manifold obtained by gluing two ideal tetrahedra according to the identification pattern shown in Figure 1.2.4 is indeed homeomorphic to the complement of the figure-eight knot. Until the end of the section, we shall denote this triangulation by $\mathcal{T}$. $\mathcal{T}$ has two edges e_1, e_2, corresponding to the single and double arrows in Figure 1.2.4, respectively. The figure also shows a labeling of the edges of the tetrahedra Δ_1, Δ_2 by shape parameters, which we shall use to write down the gluing equations. The gluing matrices for $\mathcal{T}$ are

$$G = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad G' = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \qquad G'' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, the edge consistency equations can be written as

$$e_1: \qquad z_1(z_1'')^2 z_2{}^2 z_2' = 1, \tag{1.2.9}$$

$$e_2: \qquad z_1(z_1')^2 z_2'(z_2'')^2 = 1. \tag{1.2.10}$$

In order to write down the completeness equation, we choose the peripheral curve θ_1 to be the curve μ shown in Figure 1.2.5. In this way, we have

$$G_\mu = G'_\mu = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad G''_\mu = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

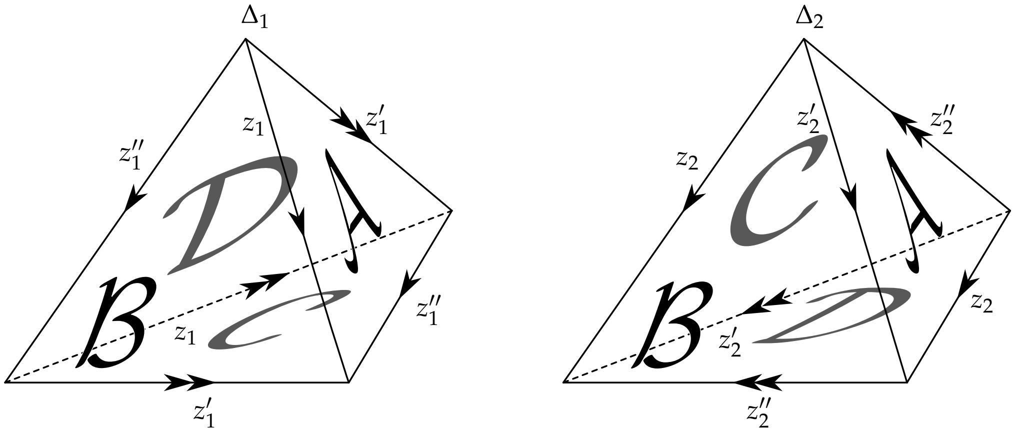


Figure 1.2.4: The identification pattern of the two-tetrahedron triangulation of the figure-eight knot complement. The corresponding faces labeled $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ should be glued in pairs in the way indicated by the arrows on the edges. The resulting triangulation has two edges, corresponding to the single and double arrows.

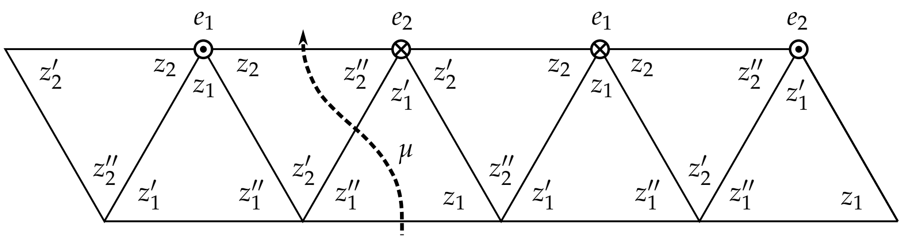


Figure 1.2.5: A fundamental domain for the link of the ideal vertex of the figure-eight knot complement, as viewed from the ideal vertex, with the triangulation into the truncation triangles (cf. Figure 1.2.2, left). The top and bottom sides of the parallelogram, as well as the left and right slanted sides, are identified in pairs to form a torus T. The edges e_1, e_2 of the triangulation $\mathcal{T}$ are transverse to the torus; lifts of their intersection points with T are marked in the upper part of the diagram. The oriented curve μ (dashed line) defines a nontrivial element of $H_1(T, \mathbb{Z})$.

and the completeness equation along μ becomes

$$\log z_1'' - \log z_2'' = 0. \tag{1.2.11}$$

Remark 1.2.7. The curve μ is a knot-theoretic meridian, which explains the notation used. In particular, μ satisfies Assumption 1.2.1.

We shall now find the unique geometric solution. Observe that in order for (1.2.11) to be satisfied, we must have $z_1'' = z_2''$, whence $z_1 = z_2 =: z$. Substituting this into the first edge equation (1.2.9), we obtain

$$z(z'')^2 z^2 z' = 1.$$

Using (1.2.2), we simplify the above equation to

$$z'' z^2 = -1, \quad \text{i.e.,} \quad (z-1)z = -1.$$

Therefore, the shape parameter solution $z = z_1 = z_2$ must be a root of the polynomial $z^2 - z + 1$. There are two conjugate roots $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, which correspond to two possible orientations of the triangulated manifold. Since we are interested in solutions in the upper halfplane P, we take

$z = \frac{1+i\sqrt{3}}{2}$. Note that the solution is unique even though we have not considered the second edge consistency equation, which is satisfied automatically (cf. Remark 1.3.2). Thus, the geometric solution is

$$z_1 = z_1' = z_1'' = z_2 = z_2' = z_2'' = \frac{1+i\sqrt{3}}{2};$$

in other words, both ideal tetrahedra will be *regular* and all truncation triangles equilateral. Hence, Figure 1.2.5 is in fact accurate geometrically.

1.2.2 Example: An exotic triangulation of the figure-eight knot complement

In this section, we wish to describe the five-tetrahedron triangulation $\mathcal{T}^{\text{ex}}$ of the figure-eight knot complement with Regina [8] signature `fLLQcccecddehqrwwn`. This triangulation was discovered via a computer search by N. Hoffman and first reported in [10, Remark 3.3].

We are going to see that although $\mathcal{T}^{\text{ex}}$ is geometric, any combinatorial 2–3 Pachner move turns $\mathcal{T}^{\text{ex}}$ into a triangulation which no longer has this property. Hence, $\mathcal{T}^{\text{ex}}$ is an isolated point of the geometric Pachner graph of ideal triangulations of the figure-eight knot complement, which justifies calling it an *exotic triangulation*. Although a complete discussion of Pachner moves [45] is beyond the scope of this work, an illustration of the 2–3 move can be found in Figure 1.2.6. Note that the 2–3 bistellar moves can be applied to *ideal* triangulations, since they do not change the number of vertices.

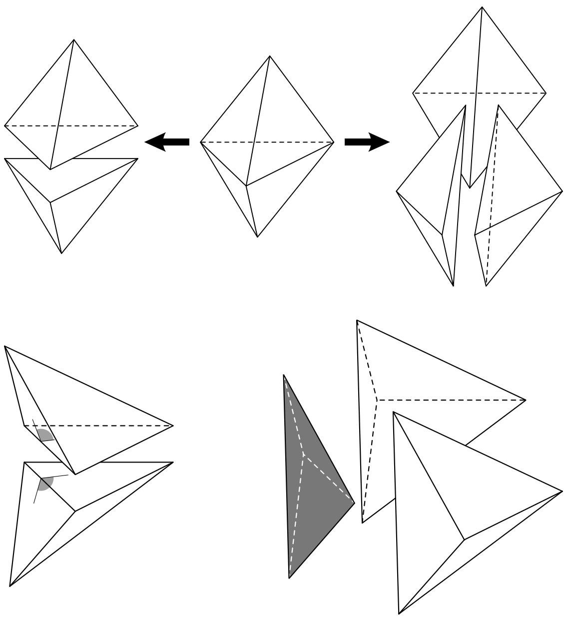


Figure 1.2.6: TOP: A 2→3 Pachner move on a triangulation consists in replacing two distinct tetrahedra glued along a face (Left) with three tetrahedra sharing a common, newly introduced edge (Right). The 3→2 Pachner move is the inverse of this operation. BOTTOM: A *non-geometric* 2→3 Pachner move on a triangulation with angles. The sum of two adjacent dihedral angles, marked on the left of the picture, exceeds π. A Pachner move is possible, but to recover the original geometry, we need the new triangulation to "fold over itself," i.e., to contain a negatively oriented tetrahedron (shaded).

The identification pattern for $\mathcal{T}^{\text{ex}}$ is shown in Figure 1.2.7. We shall use a numbering of edges which corresponds to labels in the figure as follows:

$$e_1 = \nearrow, \quad e_2 = \nearrow\!\!\!\nearrow, \quad e_3 = \nearrow\!\!\!\nearrow\!\!\!\nearrow, \quad e_4 = \nearrow, \quad e_5 = \nearrow.$$

With this numbering, and the numbering of tetrahedra shown in the figure, the gluing matrices

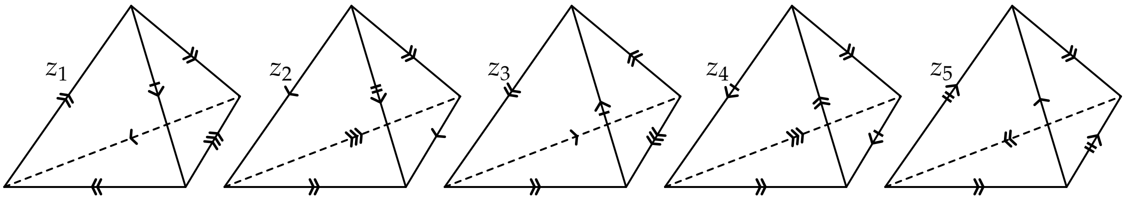


Figure 1.2.7: The gluing pattern of the five-tetrahedron triangulation of the figure-eight knot complement with Regina signature `fLLQccecddehqrwwn`. For clarity, only one edge of each tetrahedron has been labeled with its shape parameter. Remaining labels can be inferred from the rule illustrated in Figure 1.2.1, right.

for the triangulation $\mathcal{T}^{\mathrm{ex}}$ are given by

$$G = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad G' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad G'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.2.12}$$

The completeness equation can be written down with the help of the peripheral curve μ shown in Figure 1.2.8, yielding

$$\log z_3'' - \log z_1'' = 0. \tag{1.2.13}$$

The joint solution of the edge consistency and completeness equations is

$$\begin{aligned} z_1 &= z_3 = 4\omega + 2, \\ z_2 &= 3\omega + 6, \\ z_4 &= 4\omega + 1, \\ z_5 &= 3\omega - 2, \end{aligned} \qquad \text{where } \omega = \frac{1 + i\sqrt{3}}{2}, \tag{1.2.14}$$

which can be verified by direct computation. It is easy to see that the above solution is geometric, since $\operatorname{Im}\omega > 0$ and the coefficient of ω is positive in each of the shape parameter solutions found. Hence, $\mathcal{T}^{\mathrm{ex}}$ is a geometric ideal triangulation.

As a slight digression, we are now going to prove that the triangulation $\mathcal{T}^{\mathrm{ex}}$ cannot be turned into another geometric triangulation by a 2–3 Pachner move. The proof is greatly aided by Figure 1.2.8.

Proposition 1.2.8 (N. Hoffman)**.** *Any* 2→3 *or* 3→2 *Pachner move turns the triangulation $\mathcal{T}^{\mathrm{ex}}$ into a non-geometric ideal triangulation.*

Proof. We first consider 3→2 Pachner moves. As such a move can only be performed on an edge of valency 3, the only candidate in $\mathcal{T}^{\mathrm{ex}}$ is e_5. From the matrices G, G', G'' listed in (1.2.12), we see that the tetrahedra meeting at e_5 are Δ_5 (twice) and Δ_2. Therefore, the three tetrahedra meeting at e_5 are not distinct, making a 3→2 move around e_5 impossible.

Consider now all possible 2→3 Pachner moves on $\mathcal{T}^{\mathrm{ex}}$. It suffices to show that whenever F is a face of $\mathcal{T}^{\mathrm{ex}}$ along which two distinct tetrahedra Δ and Δ' meet, there exists a pair of adjacent dihedral angles of Δ and Δ' whose sum exceeds π (see Figure 1.2.6, Bottom). In other words, for every face F we need to find two shape parameters 'adjacent' along F, whose product does not have a positive imaginary part.

The faces we consider are illustrated in Figure 1.2.8, bottom, along with some of the relevant

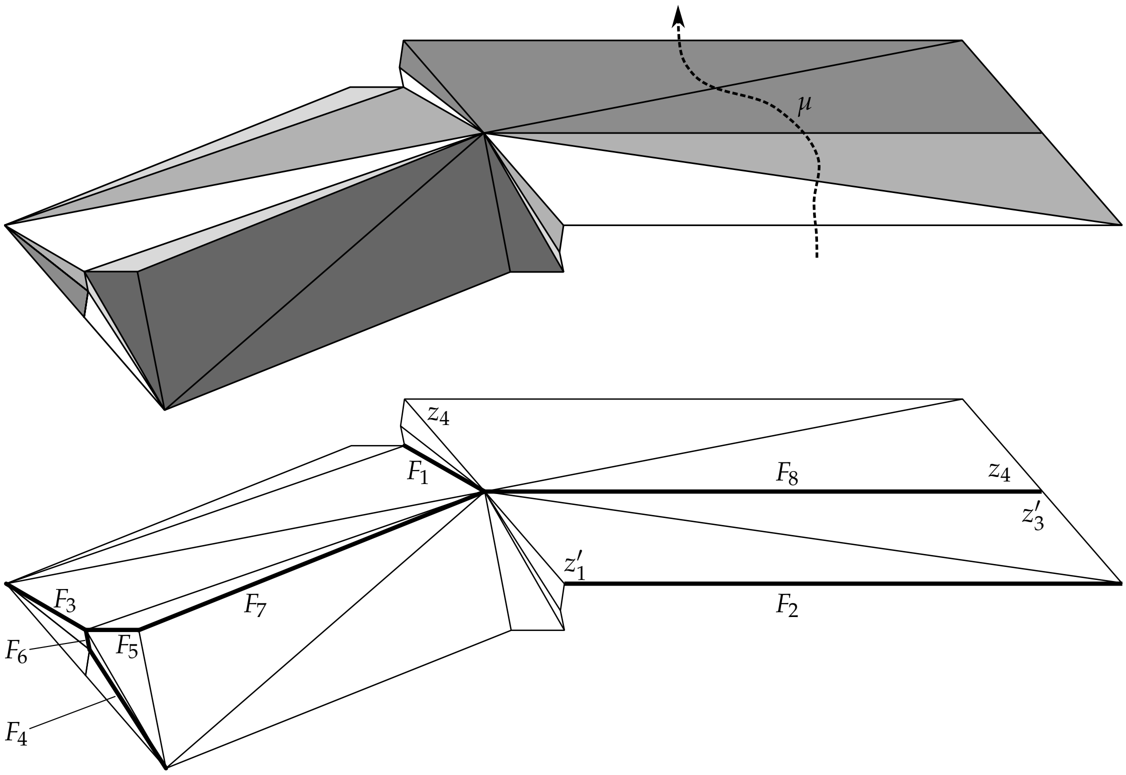


Figure 1.2.8: TOP: A fundamental domain for the cusp torus of the figure-eight knot complement obtained by gluing truncation triangles of the exotic triangulation $\mathcal{T}^{\text{ex}}$ with shape parameters listed in (1.2.14). The shading of the triangles corresponds to the numbering of the tetrahedra in Figure 1.2.7, the darkness increasing with the index of the tetrahedron. The meridian curve marked μ was used to write down the completeness equation (1.2.13). Note that the dependence of the log-holonomy along μ on the shape parameter z_4 vanishes due to the contributions from two of the truncation triangles traversed by μ cancelling each other. BOTTOM: The thickened segments correspond to faces $F_1, \dots, F_8$ of the triangulation $\mathcal{T}^{\text{ex}}$ along which the Pachner 2→3 move is possible combinatorially, but results in a non-geometric ideal triangulation.

shape parameters. Note that two further faces F_9 and F_{10} of $\mathcal{T}^{\text{ex}}$ are not shown in the figure, since they are not incident to distinct tetrahedra (the existence of self-identifications corresponds to the existence of neighbouring, equally shaded triangles in the top panel of Figure 1.2.8).

Using the relation $\omega^2 = \omega - 1$, we can easily calculate products of the following shape parameters corresponding to pairs of dihedral angles meeting along each of the faces $F_1, \dots, F_8$:

along F_1: $z_3' z_1' = \left(\frac{4\omega - 5}{21}\right)^2 = \frac{-8\omega + 3}{147}$ – negative imaginary part,

along F_2: $z_1' z_4 = \frac{4\omega - 5}{21}(4\omega + 1) = -1$ – imaginary part zero,

along F_3: $z_3' z_1'$ – same value as for F_1,

along F_4: $z_1 z_2' = (4\omega + 2)\frac{3\omega - 8}{49} = \frac{-2\omega - 4}{7}$ – negative imaginary part,

along F_5: $z_5 z_2' = (3\omega - 2)\frac{3\omega - 8}{49} = \frac{-3\omega + 1}{7}$ – negative imaginary part,

along F_6: $z_2' z_3$ – same value as for F_4 since $z_1 = z_3$,

along F_7: $z_2' z_5$ – same value as for F_5,

along F_8: $z_3' z_4$ – same value as for F_2 since $z_1' = z_3'$.

Hence, any combinatorially valid 2→3 Pachner move on $\mathcal{T}^{\text{ex}}$ results in the introduction of either a negatively oriented ideal tetrahedron or a flattened (zero volume) ideal tetrahedron. Q.E.D.

1.3 Differential properties of the gluing equations

Assume that the manifold M has a geometric ideal triangulation $\mathcal{T}$. In this section, we review certain results concerning the tangent space to the gluing variety $\mathcal{V}_{\mathcal{T}}^{+}$ at a regular point. Let $z_* \in \mathcal{V}_{\mathcal{T}}^{+}$ be the point corresponding to the unique complete hyperbolic structure on M with $k > 0$ cusps; by results of Neumann–Zagier [40], z_* is known to be a regular point of $\mathcal{V}_{\mathcal{T}}^{+}$ at which the dimension of $\mathcal{V}_{\mathcal{T}}^{+}$ is equal to k. Throughout the section, all vector spaces are over $\mathbb{C}$ and all derivatives are holomorphic derivatives.

In [9], Eun-Young Choi gave a detailed characterization of the holomorphic derivative of the map g defined in (1.2.8), which we now summarize. Fix a numbering of the cusps of M by integers $\{1, \ldots, k\}$ arbitrarily, and define

$$K_{li} = |T_l \cap e_i| \text{ for } 1 \le l \le k, \quad 1 \le i \le N, \tag{1.3.1}$$

where T_l is a horospherical torus about the lth cusp and e_i is the ith edge of $\mathcal{T}$. In other words, $K_{li} \in \{0, 1, 2\}$ counts the number of intersection points of e_i with T_l, without any regard to orientations. Taken together, the numbers K_{li} form a matrix

$$K \xlongequal{\text{def}} [K_{li}] \in \mathcal{M}_{k \times N}(\mathbb{Z}). \tag{1.3.2}$$

Remark 1.3.1. In the case of a single cusp, we have $K_{1,i} = 2$ for all i, since every edge of $\mathcal{T}$ must start and end at the sole cusp. In matrix notation, $K = [2 \ 2 \ \cdots \ 2]$. See Figure 1.2.5 for an example.

Using these purely combinatorial data, we can define a map

$$p \colon \mathbb{C}_{\times}^{N} \to \mathbb{C}_{\times}^{k}, \quad p(x_1, \ldots, x_N) = \Big(\prod_{i=1}^{N} x_i^{K_{li}} \Big)_{l=1}^{k}. \tag{1.3.3}$$

With this notation, Choi proved [9, Theorem 3.4] that for any $z \in \mathcal{V}_{\mathcal{T}}^{+}$, the sequence given by the holomorphic derivatives of g and p,

$$T_z P^N \xrightarrow{Dg} T_1 \mathbb{C}_{\times}^{N} \xrightarrow{Dp} \mathbb{C}^k \to 0,$$

is exact. Generalizing observations made in [40], Choi also shows that under the assumption that $\mathcal{T}$ is geometric, the kernel of Dg can be parametrized by derivatives of log-holonomies of arbitrarily chosen non-trivial, oriented, closed peripheral curves.

More precisely, under Assumption 1.2.1, the left-hand sides of (1.2.7) provide local holomorphic coordinates on $\mathcal{V}_{\mathcal{T}}^{+}$; these coordinates are usually written as

$$u_l = \sum_{j=1}^{N} G_{\theta_l,j} \log z_j + G'_{\theta_l,j} \log z'_j + G''_{\theta_l,j} \log z''_j, \quad 1 \le l \le k. \tag{1.3.4}$$

Note that $(u_1, \ldots, u_k) = 0$ precisely when the hyperbolic structure is complete. For our purposes, it is convenient to work with a local inverse of this parametrization,

$$y = y_\theta \colon U \to \mathcal{V}_{\mathcal{T}}^{+}, \quad y \colon (u_1, \ldots, u_k) \mapsto (z_1(u_1, \ldots, u_k), \ldots, z_N(u_1, \ldots u_k)), \tag{1.3.5}$$

where U is a neighbourhood of 0 in $\mathbb{C}^k$. In other words, y is an analytic coordinate chart on the complex manifold $\mathcal{V}_{\mathcal{T}}^{+}$. It maps U to a neighbourhood of $y(0) = z_*$, the point in $\mathcal{V}_{\mathcal{T}}^{+}$ correspoinding to the unique complete structure. We remark that in the literature of the subject, U is often called the 'generalized Dehn surgery space'; the map $y = y_\theta$ assigns to a generalized Dehn surgery parameter $u = (u_1, \ldots, u_k)$ the shape parameters of the triangulation $\mathcal{T}$ after the surgery. Note that the map y is well-defined only after the multicurve $\theta = (\theta_1, \ldots, \theta_k)$ satisfying Assumption 1.2.1 has been chosen. Nevertheless, y only depends of the homology classes of θ_l in $H_1(T_l, \mathbb{Z})$ for every l. We refer to [40, pp. 320–321] for more details on this parametrization.

Suppose that the multicurve θ is fixed arbitrarily. As a corollary of [9, Theorem 4.13], we obtain an exact sequence of complex vector spaces

$$T\mathcal{GE}\colon \qquad 0 \to T_u U \xrightarrow{Dy} T_z P^N \xrightarrow{Dg} T_1 \mathbb{C}_{\times}^N \xrightarrow{Dp} \mathbb{C}^k \to 0, \tag{1.3.6}$$

where $z = y(u)$. The symbol $T\mathcal{GE}$ refers to the fact that the above exact sequence is the result of taking the holomorphic derivative of the gluing map g. Roughly speaking, it represents the result of 'differentiating Thurston's gluing equations'. In addition, $T\mathcal{GE}$ identifies the kernel and the cokernel of Dg. $\operatorname{Ker} Dg$ is the tangent space to the complex variety $\mathcal{V}_{\mathcal{T}}^{+}$; (1.3.6) simply says that y is a holomorphic coordinate on this variety. On the other hand, the map Dp identifies $\operatorname{Coker} Dg$ with $\mathbb{C}^k$, explaining and measuring the redundancy of the edge consistency equations (1.2.3). We refer the reader to [9] for more details.

Remark 1.3.2. When $k = 1$, the redundancy of the gluing consistency equations (1.2.3) is easy to understand directly, cf. also [40]. Recall that each of the pairs of opposite edges of a tetrahedron Δ_j is assigned a shape parameter z_j, z_j', or z_j''; these parameters satisfy (1.2.2). Hence, the product of the parameters labeling all six edges of an ideal tetrahedron always equals 1. As a consequence, any collection of $N - 1$ equations (1.2.3) implies the remaining equation.

1.3.1 Calculation of Jacobians

At present, we are interested in writing down the complex linear maps occurring in (1.3.6) as explicit matrices with respect to the bases chosen as follows.

- Basis of $T_u U$: Since the coordinates on U were denoted $u_1, \ldots, u_k$, a convenient basis for $T_u U$ is given by $\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_k}\}$ at u;
- Basis of $T_z P^N$: $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}\}$ at $z = y(u)$;
- Basis of $T_1 \mathbb{C}_{\times}^N$: $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\}$, where $x_1, \ldots, x_N$ are coordinates on $\mathbb{C}_{\times}^N$ – in agreement with the notation of (1.3.3);
- Basis of the final term $\mathbb{C}^k$ is the standard basis.

We start by calculating the Jacobian matrix Jac p, i.e., the matrix of $Dp\colon T_1\mathbb{C}^N_\times \to \mathbb{C}^k$ with respect to the bases listed above. We have

$$Dp\left(\frac{\partial}{\partial x_i}\right) = \left(\frac{\partial p_l}{\partial x_i}\right)_{l=1}^k = \left(\frac{\partial}{\partial x_i}\prod_{n=1}^N x_n^{K_{ln}}\right)_{l=1}^k = \left(K_{li}x_i^{K_{li}-1}\prod_{n\neq i} x_n^{K_{ln}}\right)_{l=1}^k,$$

where the integers K_{li} were defined in (1.3.1). Setting $(x_1,\dots,x_N) = (1,\dots,1)$, we see that the sought for Jacobian matrix of Dp is simply the matrix K defined in (1.3.2):

$$\operatorname{Jac} p = K. \tag{1.3.7}$$

Working our way back through (1.3.6), we now look at $Dg\colon T_zP^N \to T_1\mathbb{C}^N_\times$. To prevent confusion, we stress that all prime symbols occurring below refer to the shape parameters given by (1.2.1); holomorphic derivatives are written exclusively in the Leibniz notation '$\frac{\partial}{\partial z}$'. The (i,j)-th entry of the Jacobian matrix of Dg at the point z_* is given by

$$[\operatorname{Jac} g]_{i,j} = \frac{\partial}{\partial z_j}\Big(\prod_{m=1}^N {z_m}^{G_{im}} {z'_m}^{G'_{im}} {z''_m}^{G''_{im}}\Big).$$

Keeping in mind the relations (1.2.1), we calculate

$$\begin{aligned}
&\frac{\partial}{\partial z_j}\Big({z_j}^{G_{ij}}{z'_j}^{G'_{ij}}{z''_j}^{G''_{ij}}\prod_{m\neq j}{z_m}^{G_{im}} {z'_m}^{G'_{im}} {z''_m}^{G''_{im}}\Big)\\
&\quad=\left(G_{ij}{z_j}^{G_{ij}-1}{z'_j}^{G'_{ij}}{z''_j}^{G''_{ij}} + G'_{ij}{z_j}^{G_{ij}}{z'_j}^{G'_{ij}-1}{z''_j}^{G''_{ij}}\frac{1}{(z_j-1)^2} + G''_{ij}{z_j}^{G_{ij}}{z'_j}^{G'_{ij}}{z''_j}^{G''_{ij}-1}\frac{1}{z_j^2}\right)\\
&\qquad\times\prod_{m\neq j}{z_m}^{G_{im}} {z'_m}^{G'_{im}} {z''_m}^{G''_{im}}\\
&\quad=\left(\frac{G_{ij}}{z_j}+\frac{G'_{ij}}{1-z_j}+\frac{G''_{ij}}{z_j(z_j-1)}\right)\underbrace{\prod_{m=1}^N{z_m}^{G_{im}} {z'_m}^{G'_{im}} {z''_m}^{G''_{im}}}_{g_i(z)}\\
&\quad=\frac{G_{ij}}{z_j}+\frac{G'_{ij}}{1-z_j}+\frac{G''_{ij}}{z_j(z_j-1)},
\end{aligned}$$

where in the last step we used the fact that $g_i(z) = 1$ for all i, which follows since $z \in \mathcal{V}^+_{\mathcal{T}}$. We have thus arrived at the formula

$$[\operatorname{Jac} g]_{i,j} = \frac{G_{ij}}{z_j}+\frac{G'_{ij}}{1-z_j}+\frac{G''_{ij}}{z_j(z_j-1)}. \tag{1.3.8}$$

Notational Convention 1.3.3. We introduce the following notation

$$\boxed{\zeta_j = \frac{1}{z_j},\quad \zeta'_j = \frac{1}{1-z_j},\quad \zeta''_j = \frac{1}{z_j(z_j-1)}} \tag{1.3.9}$$

for $1 \le j \le N$. Furthermore, we write $\zeta = (\zeta_1,\dots,\zeta_N)$, $\zeta' = (\zeta'_1,\dots,\zeta'_N)$, $\zeta'' = (\zeta''_1,\dots,\zeta''_N)$.

Using the above notation and diagonal matrices, we can rewrite (1.3.8) more simply as

$$\operatorname{Jac} g = G \operatorname{diag}(\zeta) + G' \operatorname{diag}(\zeta') + G'' \operatorname{diag}(\zeta''). \tag{1.3.10}$$

Lastly, we consider the map $Dy\colon T_uU \to T_zP^N$. We assume that the integers $G_{\theta_l,j}$, $G'_{\theta_l,j}$ and $G''_{\theta_l,j}$ are exactly those appearing in equation (1.3.4) which we repeat below.

$$u_l = \sum_{j=1}^{N} G_{\theta_l,j} \log z_j + G'_{\theta_l,j} \log z'_j + G''_{\theta_l,j} \log z''_j, \quad 1 \le l \le k. \tag{1.3.4}$$

It is convenient to organize these integers into $k \times N$ matrices

$$G_\theta = \left[G_{\theta_l,j}\right]_{l,j}, \quad G'_\theta = \left[G'_{\theta_l,j}\right]_{l,j}, \quad G''_\theta = \left[G''_{\theta_l,j}\right]_{l,j}. \tag{1.3.11}$$

Taking any $n \in \{1,\dots,k\}$ and differentiating (1.3.4) with respect to u_n, we obtain

$$\delta_{l,n} = \sum_{j=1}^{N} \left(G_{\theta_l,j} \frac{\partial \log z_j}{\partial z_j} + G'_{\theta_l,j} \frac{\partial \log z'_j}{\partial z_j} + G''_{\theta_l,j} \frac{\partial \log z''_j}{\partial z_j} \right) \frac{\partial z_j}{\partial u_n}, \tag{1.3.12}$$

where $\delta_{l,n}$ is the Kronecker symbol. Furthermore, we find

$$\frac{\partial \log z_j}{\partial z_j} = \frac{1}{z_j} = \zeta_j, \qquad \frac{\partial \log z'_j}{\partial z_j} = (1-z_j)\frac{\partial \frac{1}{1-z_j}}{\partial z_j} = \frac{1}{1-z_j} = \zeta'_j,$$
$$\frac{\partial \log z''_j}{\partial z_j} = \frac{z_j}{z_j-1}\frac{\partial}{\partial z_j}\left(1-\frac{1}{z_j}\right) = \frac{1}{z_j(z_j-1)} = \zeta''_j, \tag{1.3.13}$$

which allows us to write (1.3.12) as

$$\delta_{l,n} = \sum_{j=1}^{N} \left(G_{\theta_l,j}\zeta_j + G'_{\theta_l,j}\zeta'_j + G''_{\theta_l,j}\zeta''_j \right) \frac{\partial z_j}{\partial u_n}. \tag{1.3.14}$$

The Jacobian matrix of the map y at $u \in U$ is, by definition, $\operatorname{Jac} y = \left[\frac{\partial z_j}{\partial u_n}\right]_{j,n} \in \mathcal{M}_{N\times k}(\mathbb{C})$, where $z = y(u)$. Using matrix notation, we can express (1.3.14) more simply as

$$\operatorname{Id}_{k\times k} = \left(G_\theta \operatorname{diag}(\zeta) + G'_\theta \operatorname{diag}(\zeta') + G''_\theta \operatorname{diag}(\zeta'')\right) \operatorname{Jac} y. \tag{1.3.15}$$

Although we are not able to find an explicit expression for $\operatorname{Jac} y$, we have shown that the explicit matrix $G_\theta \operatorname{diag}(\zeta) + G'_\theta \operatorname{diag}(\zeta') + G''_\theta \operatorname{diag}(\zeta'')$ is a left inverse of $\operatorname{Jac} y$. This is expected, since we started with explicit expressions for log-parameters u_l given in (1.3.4) and took y to be the local inverse of this explicit parametrization.

Remark 1.3.4. Matrix notations similar to those used above were introduced by Neumann and Zagier in [40]. Therein, on p. 315, one can find a computation of the Jacobian of Thurston's gluing equations similar to the calculations presented here, one minor difference being that Neumann and Zagier eliminated one of the three shape parameters for each tetrahedron using the relation (1.2.2). In our notation, this relation implies the identity $\zeta_j + \zeta'_j + \zeta''_j = 0$, which can be verified directly from (1.3.9).

Chapter 2

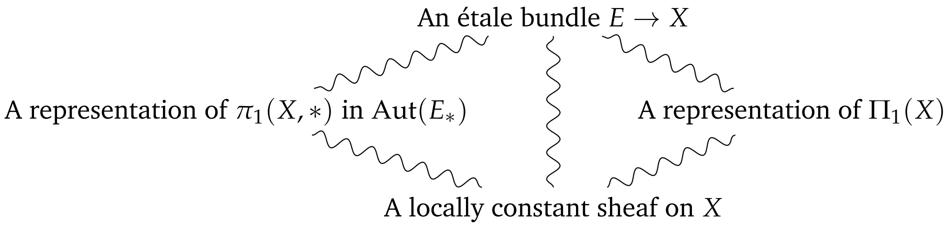
Holonomy representations of hyperbolic structures

In this chapter, we describe how to associate to every hyperbolic structure on a 3-manifold M a representation $\varrho\colon \pi_1(M) \to PSL_2\mathbb{C}$ called the *holonomy representation* of the hyperbolic structure. This representation ϱ is not defined uniquely, but rather only up to conjugation. In particular, we are interested in understanding the different ways in which a representative of the conjugacy class of ϱ can be constructed from a geometric ideal triangulation of M together with a shape parameter solution of Thurston's gluing equations.

There are several well established ways of thinking about the holonomy of a hyperbolic structure. Perhaps the easiest way is not to consider ϱ at all, but instead to define the hyperbolic manifold M as the quotient $\mathbb{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group. This top-down approach to hyperbolic manifolds, ultimately rooted in the Erlangen programme, is used for instance in [33]. In contrast, Thurston's description of holonomy representations in [50] relies on the concept of a *developing map* and could be called a 'bottom-up' approach in that it does not assume that M is given a priori by a Kleinian group Γ, but rather attempts to reconstruct Γ from the knowledge of M together with its geometry.

In this largely expository chapter, we are going to present an approach similar to that of A. Haefliger [23] and understand the 'hyperbolic structure' on M in terms of a certain sheaf on M, which we shall call the *sheaf of germs of developing maps*. By considering germs of developing maps at different points of the manifold, we shall provide a groupoid description of the holonomy representation, of which the group homomorphism $\varrho\colon \pi_1(M) \to PSL_2\mathbb{C}$ will be a special case. We believe that this approach deserves to be better known, as it greatly simplifies the study of holonomy representations of hyperbolic 3-manifolds triangulated into ideal tetrahedra.

The relationships between the different approaches mentioned above can be described, in a much greater generality, as follows. Let X be a path-connected topological space with a distinguished basepoint $* \in X$. A locally trivial fibre bundle $E \xrightarrow{\pi} X$ is called *étale* if π is a local homeomorphism, i.e., every point of E has an open neighbourhood $U \subset E$ such that $\pi|_U\colon U \to \pi(U)$ is a homeomorphism. There are many equivalent ways of thinking about étale bundles, as illustrated by the philosophical diagram below.



We briefly outline some of the relationships represented by the wavy lines in the diagram.

- The vertical line corresponds to the fact that every sheaf has an associated *étale space* and vice versa: every sheaf is isomorphic to the sheaf of continuous sections of an étale map. In particular, étale *bundles* correspond to locally constant sheaves. We refer to Godement [20, Section II.1.2] for the details of this construction.

- An étale bundle $E \to X$ determines a representation of the fundamental groupoid $\Pi_1(X)$ in the full subcategory of Set containing the fibres of E as objects. This functor sends a point $x \in X$ to the fibre E_x and a homotopy class of a path from x to y to the unique map $E_x \to E_y$ constructed by lifting the path to E.

- A locally constant sheaf induces a monodromy representation in the group of automorphisms of its stalk at the basepoint and vice versa [49, Theorem 1].

In this chapter, we are going to see how some of these general relationships manifest themselves in the context of hyperbolic structures on 3-manifolds. Throughout the chapter, we assume M is a connected, oriented 3-manifold equipped with a complete hyperbolic structure of finite volume. The corresponding representation of the fundamental group $\pi_1(M, *)$ in $\mathrm{Isom}^+(\mathbb{H}^3)$, known in the literature as the *holonomy representation* of the hyperbolic structure, will correspond to the leftmost vertex of our conceptual diagram. The locally constant sheaf at the bottom is what we shall call the *sheaf of germs of developing maps* and discuss in Section 2.2.

The term *holonomy* is used in differential geometry in a much broader context, which we briefly recall. Suppose X is a smooth manifold and $E \to X$ is a smooth fibre bundle with a connection ∇. For an arbitrary $x \in X$ and a smooth path $\alpha\colon [0,1] \to X$ with $\alpha(0) = \alpha(1) = x$, we can consider the *parallel transport* of an element $v \in E_x$ along the path α. The parallel transport is defined in terms of ∇-parallel lifts of the path α to the total space E. Since there is a unique parallel lift $\widetilde{\alpha}$ satisfying $\widetilde{\alpha}(0) = v$, the endpoint $\mathrm{PT}_\alpha(v) := \widetilde{\alpha}(1) \in E_x$ is well-defined and is called the parallel transport of v along α. The map $\mathrm{PT}_\alpha\colon E_x \to E_x$ is then called the holonomy of E along the path α. More generally, we can consider the map $\mathrm{PT}\colon \mathrm{Pth}_x \to \mathrm{Aut}(E_x)$ defined by $\mathrm{PT}\colon \alpha \mapsto \mathrm{PT}_\alpha$, where Pth_x is the set of all smooth paths in X starting and ending at x.

In general, the holonomy $\mathrm{PT}\colon \mathrm{Pth}_x \to \mathrm{Aut}(E_x)$ is not invariant under homotopy of paths. However, in the special case of a *flat* bundle (a bundle with a connection ∇ of vanishing curvature), the parallel transport does not change under homotopy of paths. In this case, the map PT descends to the fundamental group $\pi_1(X, x) = \mathrm{Pth}_x /{\simeq}$, defining a homomorphism $\mathrm{PT}\colon \pi_1(X, x) \to \mathrm{Aut}(E_x)^{\mathrm{op}}$. Here, the superscript ‘op’ emphasizes that the order of composition of automorphisms of the fibre must be opposite to the standard convention on composition of functions and correspond instead to writing function evaluation in the post-fix notation:

$$(v)(f \circ g) = ((v)f)\, g,$$

since this is the order in which paths representing elements of the fundamental group are traversed. To avoid such notational oddities, it is customary to take inverses, i.e., to compose PT with the map

$$(\cdot)^{-1}\colon \operatorname{Aut}(E_x)^{\mathrm{op}} \xrightarrow{\cong} \operatorname{Aut}(E_x) \qquad (2.0.1)$$
$$g \mapsto g^{-1}.$$

In the situations of greatest interest to us, the holonomy representations of flat bundles can often be constructed purely topologically, for example by using locally constant sheaves or, equivalently, the associated étale bundles. Further, these topological constructions are often more elementary and fundamental than their interpretation as flat connections. For this and other reasons, it would perhaps be more appropriate to use the term *monodromy* representation of the hyperbolic structure. The word *monodromy*, from Greek μόνος *only, sole* and δρόμος *way, path*, emphasizes the homotopy invariance of the parallel lift of a path. Nevertheless, it has become customary in the literature on hyperbolic geometry to always use the term 'holonomy representation' and we shall adhere to this custom. However, we do use the term 'monodromy' when talking about locally constant sheaves. We hope this usage does not lead to confusion.

In a sense, the two groupoids $\Pi_1(X)$ and $\pi_1(X,*)$ lie on the opposite ends of a spectrum: every point of X is an object in $\Pi_1(X)$, whereas $\pi_1(X,*)$ has only one object $*$. Nevertheless, $\Pi_1(X)$ and $\pi_1(X,*)$ are equivalent. Therefore, one may also consider an intermediate groupoid, for example a finitely generated subgroupoid of $\Pi_1(X)$ given by a finite good open cover of X. Our interest in groupoid descriptions of holonomy representations stems from the fact that a geometric ideal triangulation of a hyperbolic 3-manifold gives a natural collecion of basepoints for such an intermediate groupoid. This idea was applied to the description of moduli spaces of local systems by Fock and Goncharov [17]. In [12, Appendix B], Tudor Dimofte and Roland van der Veen explain how to carry out the same construction on a triangulated 3-manifold. The entire Section 2.3 is devoted to the study of the holonomy representations of triangulated oriented hyperbolic 3-manifolds from the perspective of the groupoids associated to ideal triangulations.

2.1 Holonomy representations and developing maps

We wish to recall the definition of the *holonomy representation* of the fundamental group of a hyperbolic 3-manifold M. Our discussion is based on the Princeton lecture notes by W. Thurston [50]. Since we are interested in properties of geometric ideal triangulations, it is sufficient to restrict our attention to the case of orientable hyperbolic 3-manifolds.

Suppose $* \in M$ is an arbitrarily chosen point in the connected, oriented hyperbolic 3-manifold M. Suppose also that $U_1 \subset M$ is an open neighbourhood of $*$ and $\varphi_1\colon U_1 \to \mathbb{H}^3$ is an orientation-preserving geometric chart. Note that any two such charts differ by a composition with an orientation-preserving isometry of $\mathbb{H}^3$. On the other hand, every isometry of $\mathbb{H}^3$ is a real-analytic map, hence fully determined by its restriction to any nonempty open set. This leads us to the study of the notion of analytic continuation of geometric charts on M.

Given a path $\alpha\colon [0,1] \to M$ with $\alpha(0) = *$, we can cover $\alpha([0,1])$ with open domains U_i of geometric charts φ_i where $i \in \{1,\ldots,n\}$ for some $n > 1$. In other words, $\varphi_1\colon U_1 \to \mathbb{H}^3$ is the original chart near $*$ and we assume the charts $\varphi_i\colon U_i \to \mathbb{H}^3$ are indexed in such a way that $U_i \cap U_{i+1} \neq \emptyset$ for all $i \in \{1,\ldots,n-1\}$. For simplicity, we may require that U_i be connected and simply connected for all i. Since $\varphi_{i+1}|_{U_i \cap U_{i+1}} \circ \varphi_i^{-1} \colon \varphi_i(U_i) \to \mathbb{H}^3$ is an orientation-preserving local isometry of $\mathbb{H}^3$, it extends to a unique global isometry $g_i \in \mathrm{Isom}^+(\mathbb{H}^3)$. Furthermore, φ_i and $g_i^{-1} \circ \varphi_{i+1}$ agree on $U_i \cap U_{i+1}$, so we can regard $g_i^{-1} \circ \varphi_{i+1}$ as the analytic continuation of φ_i onto U_{i+1}. Continuing this process all the way along α, we see that the final open set $U_n \ni \alpha(1)$ carries a distinguished chart $c_{\alpha,\varphi_1} := (g_1 \circ \cdots \circ g_{n-1})^{-1} \circ \varphi_n$ which is the unique analytic continuation of the initial chart φ_1 along α. In particular, the image $c_{\alpha,\varphi_1}(\alpha(1)) \in \mathbb{H}^3$ is well-defined (see Figure 2.1.1). Since $\mathbb{H}^3$ is simply-connected, the analytic continuation of geometric charts does not change under homotopy of paths relative to their endpoints. Therefore, $c_{\alpha,\varphi_1}(\alpha(1))$ only depends on the homotopy class of α.

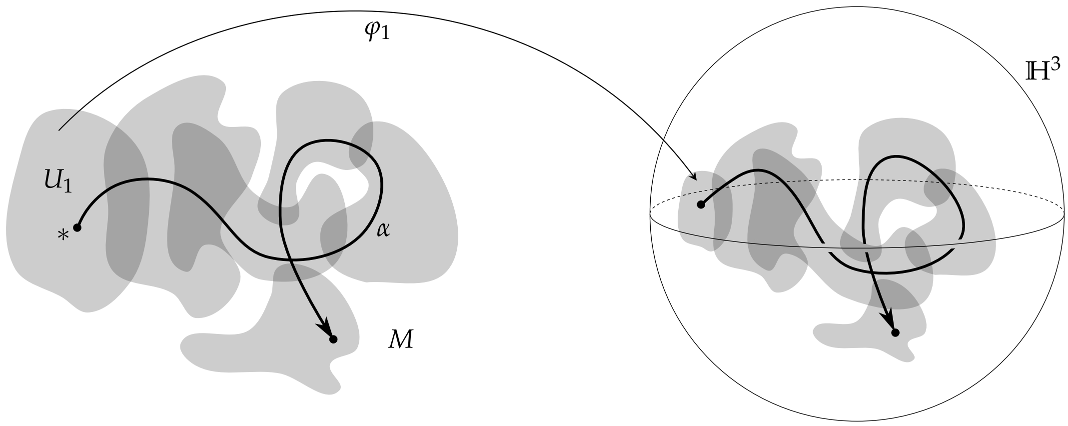


Figure 2.1.1: A local geometric chart φ_1 on a neighbourhood U_1 of $*$ can be analytically continued in a unique way along any path α in M starting at $*$. Since $\mathbb{H}^3$ is simply connected, this analytic continuation does not change under homotopy of α relative to its endpoints.

The notion of analytic continuation of geometric charts can be used to define the *holonomy representation* of $\pi_1(M,*)$ in $\mathrm{Isom}^+(\mathbb{H}^3)$. Suppose that $\alpha \colon [0,1] \to M$ is a closed path with $\alpha(0) = \alpha(1) = *$. As before, the geometric chart φ_1 can be uniquely continued along α, yielding a new geometric chart $c_{\alpha,\varphi_1} \colon U_n \to \mathbb{H}^3$. This time, however, the intersection $U_1 \cap U_n$ is nonempty, so the continued chart c_{α,φ_1} can be compared to the original chart φ_1. Since all charts were orientation-preserving, there exists a unique element $g_\alpha \in \mathrm{Isom}^+ \mathbb{H}^3$ such that

$$\varphi_1 = g_\alpha \circ c_{\alpha,\varphi_1} \text{ on } U_1 \cap U_n. \tag{2.1.1}$$

This element g_α only depends on the homotopy class of α. If β is another closed path based at $*$, we can consider the composite path $\alpha \cdot \beta$. Denote by χ the chart c_{α,φ_1}. We have

$$\begin{aligned} g_{\alpha\cdot\beta} \circ c_{\beta,\chi} &= g_{\alpha\cdot\beta} \circ c_{\alpha\cdot\beta,\varphi_1} \\ &= \varphi_1 = (\varphi_1 \circ \chi^{-1}) \circ \chi \\ &= (\varphi_1 \circ \chi^{-1}) \circ (g_\beta \circ c_{\beta,\chi}) = (g_\alpha \circ c_{\alpha,\varphi_1} \circ \chi^{-1}) \circ g_\beta \circ c_{\beta,\chi} \\ &= g_\alpha \circ g_\beta \circ c_{\beta,\chi}. \end{aligned}$$

Hence, we must have $g_{\alpha\cdot\beta} = g_\alpha \circ g_\beta$. This proves that the assignment

$$\begin{aligned} \mathrm{hol}\colon \pi_1(M,*) &\to \mathrm{Isom}^+(\mathbb{H}^3) \\ [\alpha] &\mapsto g_\alpha \end{aligned} \tag{2.1.2}$$

is a well-defined group homomorphism, provided that the initial geometric chart φ_1 is fixed.

Remark 2.1.1. Without a fixed initial geometric chart φ_1, the holonomy representation of (2.1.2) is well defined up to conjugation only. To see this, suppose φ_1, φ_1' are two initial charts and consider the two corresponding homomorphisms

$$\mathrm{hol}, \mathrm{hol}'\colon \pi_1(M,*) \to \mathrm{Isom}^+(\mathbb{H}^3).$$

Since there exists a unique element $g \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $\varphi_1' = g\varphi_1$, we have

$$\mathrm{hol}'(\alpha)c_{\alpha,\varphi_1'} = \varphi_1' = g\varphi_1 = g\,\mathrm{hol}(\alpha)c_{\alpha,\varphi_1} = g\,\mathrm{hol}(\alpha)c_{\alpha,g^{-1}\varphi_1'} = g\,\mathrm{hol}(\alpha)g^{-1}c_{\alpha,\varphi_1'},$$

showing that $\mathrm{hol}'(\,\cdot\,) = g\,\mathrm{hol}(\,\cdot\,)g^{-1}$. Therefore, any two holonomy representations are conjugate.

Definition 2.1.2. The map hol of (2.1.2) is called the *holonomy representation* of the hyperbolic structure on M. It is only defined up to conjugation.

Another way to understand the holonomy representation is via the *developing map*. The developing map can be thought of as a maximally prolongated (multi-valued) geometric coordinate chart. As before, we shall only consider orientation-preserving charts. To define the developing map, it is useful to treat the universal covering space $\widetilde{M}$ as the set of homotopy classes of paths in M starting at $*$, relative to their endpoints. Given an initial chart φ_1 on a neighbourhood of $*$, we can associate to $[\alpha] \in \widetilde{M}$ the point $c_{\alpha,\varphi_1}(\alpha(1)) \in \mathbb{H}^3$ which, as we saw above, only depends on the homotopy class of α. This defines the *developing map*

$$\begin{aligned} \mathrm{Dev}\colon \widetilde{M} &\to \mathbb{H}^3 \\ [\alpha] &\mapsto c_{\alpha,\varphi_1}(\alpha(1)). \end{aligned} \tag{2.1.3}$$

Note that the developing map is not unique, since it depends on the choice of the initial geometric chart φ_1. Hence, the indeterminacy of the developing map is the same as the indeterminacy in the definition of the holonomy representation.

The promised relationship between the developing map and the holonomy representation is as follows. Suppose that a basepoint $* \in M$ is fixed and that $\varphi_1\colon U_1 \to \mathbb{H}^3$ is a fixed initial geometric chart on a neighbourhood U_1 of $*$. For an element $\gamma \in \pi_1(M,*)$, denote by T_γ the corresponding deck transformation of the universal covering space $\widetilde{M}$. Then we have

$$\mathrm{Dev} \circ T_\gamma = \mathrm{hol}(\gamma) \circ \mathrm{Dev}.$$

This equality is in fact used by Thurston [50] as the definition of the holonomy representation. We remark that the same approach is possible for other (X, G)-manifolds, cf. [47].

2.2 The sheaf of germs of developing maps

In this section, we study the *sheaf of germs of developing maps*. Although this elementary concept seems to be well known to experts, we have not been able to find an authoritative source where it is discussed at any length. Similar ideas have been discussed by Guruprasad–Haefliger [23], cf. especially Section 2.1.4, and possibly in earlier work of Haefliger cited therein. Since we are assuming that M is oriented, we focus entirely on orientation-preserving developing maps and avoid using the notion of an *étale groupoid*, replacing it in Section 2.3 with the more established concept of a *groupoid representation* of the fundamental groupoid.

Recall that a real-analytic map on a connected, simply connected domain is fully determined by its values on an arbitrarily small open neighbourhood of point, hence by its *germ* at that point. It turns out that germs of local isometries can be parametrized by their values and their first derivatives only. More precisely, the following elementary fact can be found i.a. in the book by Benedetti and Petronio [4].

Proposition 2.2.1 (Proposition A.2.1 in [4]). *Let X and Y be arbitrary connected Riemannian manifolds of equal dimension. If $f, g\colon X \to Y$ are local isometries satisfying $f(x) = g(x)$ and $Df_x = Dg_x$ for some $x \in X$, then $f = g$ everywhere.*

Definition 2.2.2. For an open set $U \subseteq M$, denote by $\mathscr{D}(U)$ the set of all orientation-preserving local isometries $U \to \mathbb{H}^3$. The assignment $U \mapsto \mathscr{D}(U)$ defines a sheaf $\mathscr{D}$ called the *sheaf of germs of developing maps* on M.

The name 'sheaf of germs of developing maps' can be motivated as follows. Fix $x \in M$ arbitrarily and consider the stalk $\mathscr{D}_x$. Given any germ $\varphi \in \mathscr{D}_x$, we can use x as the basepoint and obtain a developing map $\mathrm{Dev}_\varphi\colon \widetilde{M} \to \mathbb{H}^3$ by setting $\varphi_1 = \varphi$ in (2.1.3). Therefore, stalks of the sheaf $\mathscr{D}$ can be thought of as collections of germs of orientation-preserving developing maps.

Every point $x \in M$ has an open, connected and simply connected neighbourhood U equipped with a geometric atlas $\varphi\colon U \to \mathbb{H}^3$. If U is any such neighbourhood, then $\mathscr{D}\big|_U$ is a constant sheaf on U. Hence, the sheaf $\mathscr{D}$ is locally constant.

The group $\mathrm{Isom}^+(\mathbb{H}^3)$ acts on $\mathscr{D}$ on the left; this action

$$\mathrm{Isom}^+(\mathbb{H}^3) \times \mathscr{D} \to \mathscr{D}$$

is defined as follows: for any $\varphi \in \mathscr{D}(U)$ and $g \in \mathrm{Isom}^+(\mathbb{H}^3)$, the composition $g \circ \varphi$ is a new element of $\mathscr{D}(U)$. It is obvious that this action commutes with the restriction maps of $\mathscr{D}$.

Lemma 2.2.3. *Assume that a basepoint $* \in M$ is arbitrarily fixed.*

(i) The left action of $\mathrm{Isom}^+(\mathbb{H}^3)$ *on the stalk $\mathscr{D}_*$ is free and transitive.*

(ii) Let $\mu\colon \pi_1(M, *) \to \mathrm{Aut}(\mathscr{D}_*)$ be the monodromy of $\mathscr{D}$. Moreover, choose a fixed germ $\varphi_1 \in \mathscr{D}_*$. Then

$$\mu(\gamma)\,(g \circ \varphi_1) = \mathrm{hol}(\gamma) g \circ \varphi_1 \qquad \text{for every } \gamma \in \pi_1(M, *),\ g \in \mathrm{Isom}^+(\mathbb{H}^3),$$

where hol *denotes the holonomy representation* (2.1.2).

Proof. Given a point $p \in \mathbb{H}^3$, a *dreibein* at p is an ordered, positively oriented orthonormal basis of the tangent space $T_p\mathbb{H}^3$. Choose any $p, p' \in \mathbb{H}^3$ and let d, d' be arbitrary dreibeins at p and

p', respectively. Since $\mathbb{H}^3$ is homogenous and isotropic, there exists a unique hyperbolic isometry $g \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $g(p) = p'$ and $Dg(d) = d'$. Proposition 2.2.1 now implies that the action of $\mathrm{Isom}^+(\mathbb{H}^3)$ is transitive on $\mathscr{D}_*$. To see that this action is free, consider an element $g \in \mathrm{Isom}^+(\mathbb{H}^3)$ satisfying $g \circ \varphi = \varphi$ for some germ $\varphi \in \mathscr{D}_*$. Defining $p := \varphi(*) \in \mathbb{H}^3$, we see that $g(p) = p$ and $Dg_p = \mathrm{Id}\colon T_p\mathbb{H}^3 \to T_p\mathbb{H}^3$. Hence, Proposition 2.2.1 implies that g is the identity element of $\mathrm{Isom}^+(\mathbb{H}^3)$, which finishes the proof of Part (i).

To prove Part (ii), fix a germ of a local isometry $\varphi_1 \in \mathscr{D}_*$ and denote by hol the holonomy representation defined in terms of φ_1. Note that Part (i) implies that every element of $\mathscr{D}_*$ can be written as $g \circ \varphi_1$ for a unique $g \in \mathrm{Isom}^+(\mathbb{H}^3)$. Let $\alpha\colon [0,1] \to M$ be a path starting and ending at the basepoint $*$. Recall that the construction of $\mathrm{hol}([\alpha])$ required covering $\alpha([0,1])$ with geometric coordinate charts $U_1, \dots, U_n$ (cf. Figure 2.1.1). Since the sets U_i, $1 \le i \le n$ are connected and simply connected, the sheaf $\mathscr{D}$ is trivial on each one of them. Analytic continuation of geometric charts now amounts to the unique prolongation of an initial section $\varphi_1 \in \mathscr{D}(U_1)$ along α. Thus, the monodromy map

$$\mu([\alpha])\colon \mathscr{D}_* \to \mathscr{D}_*$$

satisfies $\mu([\alpha])^{-1}(\varphi_1) = (c_{\alpha,\varphi_1})_*$ where c_{α,φ_1} is the chart occurring in equation (2.1.1). (The inverse is necessary if we want $\pi_1(M,*)$ to act on $\mathscr{D}_*$ on the *left*; see (2.0.1).) Hence,

$$\mathrm{hol}([\alpha]) \circ \mu([\alpha])^{-1}(g \circ \varphi_1) = \mathrm{hol}([\alpha]) \circ c_{\alpha, g\circ\varphi_1} = g \circ \varphi_1,$$

where the last equality is simply (2.1.1) with φ_1 replaced by $g \circ \varphi_1$. Q.E.D.

Part (ii) of the above lemma states that the holonomy of the hyperbolic structure on M is essentially equal to the monodromy of the sheaf $\mathscr{D}$. Nevertheless, the initial germ $\varphi_1 \in \mathscr{D}_*$ needs to be chosen in order to identify $\mathrm{Isom}^+(\mathbb{H}^3)$ (where hol takes its values) with $\mathscr{D}_*$. This identification is possible thanks to Part (i), which says that there is a bijective correspondence between the group $\mathrm{Isom}^+(\mathbb{H}^3)$ and the orbit of φ_1, and that this orbit includes all of $\mathscr{D}_*$. In particular, the identification of $\mathscr{D}_*$ with the orbit of φ_1 identifies $\mathrm{Aut}(\mathscr{D}_*)$ with $\mathrm{Isom}^+(\mathbb{H}^3)$.

Remark 2.2.4. Since $\mathrm{Isom}^+(\mathbb{H}^3) = PSL_2\mathbb{C}$, the étale space of the sheaf $\mathscr{D}$ is a principal $PSL_2\mathbb{C}$-bundle. This bundle can be constructed explicitly as follows. Denote by $PSL_2\mathbb{C}^\delta$ the group $PSL_2\mathbb{C}$ equipped with the discrete topology and let hol be a holonomy representation of $\pi_1(M,*)$. Then the quotient space

$$E = \widetilde{M} \times PSL_2\mathbb{C}^\delta \Big/ (T_\gamma x, \mathrm{hol}(\gamma)g) \sim (x,g), \quad \gamma \in \pi_1(M,*)$$

is the total space of a topological principal bundle on M with the property that the sheaf of continuous sections of $E \xrightarrow{\pi} M$ is isomorphic to $\mathscr{D}$.

We wish to use elements of $PSL_2\mathbb{C}$ to describe monodromy in $\mathscr{D}$ along paths connecting different points of M. Suppose $x, y \in M$ and let $\varphi_x \in \mathscr{D}_x$, $\varphi_y \in \mathscr{D}_y$ be chosen arbitrarily. For any path $\alpha\colon [0,1] \to M$ satisfying $\alpha(0) = x$, $\alpha(1) = y$, the monodromy image c_{α,φ_x} lies in $\mathscr{D}_y$, hence in the orbit of φ_y under the action of $\mathrm{Isom}^+(\mathbb{H}^3)$. In other words, there exists a unique element $g_\alpha \in PSL_2\mathbb{C}$ such that

$$c_{\alpha,\varphi_x} = g_\alpha^{-1} \circ \varphi_y. \tag{2.2.1}$$

The appearance of the inverse in the above equation is needed to ensure compatibility with

(2.1.1), since we want the monodromy maps $c_{\alpha,-}$ to act on their arguments on the left.

Definition 2.2.5.

1. Let $S \subset M$ be a finite subset. A *framing of S* is any collection of germs of developing maps at all points of S, i.e., a family $\{\varphi_s\}_{s\in S}$ satisfying $\varphi_s \in \mathscr{D}_s$ for all s.

2. Suppose $S \subset M$ is equipped with a framing $\{\varphi_s\}_{s\in S}$. Given any $x, y \in S$ and a path α in M from x to y, the unique element $g_\alpha \in PSL_2\mathbb{C}$ satisfying (2.2.1) is called the *partial monodromy along α with respect to the framing $\{\varphi_s\}$*.

Using the above terminology, Lemma 2.2.3 implies that the holonomy of the hyperbolic structure on M is equal to the partial monodromy of $\mathscr{D}$ with $S = \{*\}$ and the framing $\varphi_* = \varphi_1$.

2.3 Groupoid representations

Recall that a *groupoid* is a category in which every morphism is an isomorphism. In particular, every classical group Γ can be treated as a groupoid with a single object $*$ and $\mathrm{Hom}(*,*) = \Gamma$, with the composition law given by the group operation in Γ.

Definition 2.3.1 (Groupoid representation)**.** Let $\mathcal{G}$ be a groupoid and let $\mathcal{C}$ be any category. A representation of $\mathcal{G}$ in $\mathcal{C}$ is a (covariant) functor $\mathcal{G} \to \mathcal{C}$.

In this section, we are interested exclusively in representations of the fundamental groupoid $\Pi_1(M)$. Although most of the ideas discussed here could be presented at a very high level of generality, our primary goal is the description of the monodromy of the the sheaf $\mathscr{D}$ of germs of developing maps on M, so we shall focus on this particular case, keeping in mind that any locally constant sheaf on M can be treated analogously.

The sheaf $\mathscr{D}$ gives rise to a groupoid representation

$$\begin{aligned}\mathcal{H}o\mathcal{L}\colon \Pi_1(M) &\to \mathsf{Set}\\ \mathcal{H}o\mathcal{L}\colon x &\mapsto \mathscr{D}_x\\ \mathcal{H}o\mathcal{L}\colon (\alpha\colon x\to y) &\mapsto (c_{\alpha,-}: \mathscr{D}_x \to \mathscr{D}_y).\end{aligned}$$

As in the previous section, $c_{\alpha,-}: \mathscr{D}_x \to \mathscr{D}_y$ stands for the analytic continuation of germs of developing maps along any path homotopic to α rel. $\{x, y\}$. We shall now study the groupoid representation $\mathcal{H}o\mathcal{L}$ in more detail.

2.3.1 The groupoid associated to an ideal triangulation

Suppose $\mathcal{T}$ is a geometric ideal triangulation of M. We are going to define a sub-groupoid of $\Pi_1(M)$ associated to $\mathcal{T}$. The following lemma is elementary.

Lemma 2.3.2. *Let A and B be two ideal hyperbolic triangles in $\mathbb{H}^3$. Then there are exactly six hyperbolic isometries $g \in \mathrm{Isom}^+(\mathbb{H}^3)$ taking A to B. These isometries form an orbit under the action of the dihedral group D_3 on either of the ideal triangles.*

Proof. Every two ideal hyperbolic triangles are congruent, since the action of $PSL_2\mathbb{C}$ on $\mathbb{C}P^1$ is transitive on triples. Therefore, without loss of generality, we may assume that A is the ideal

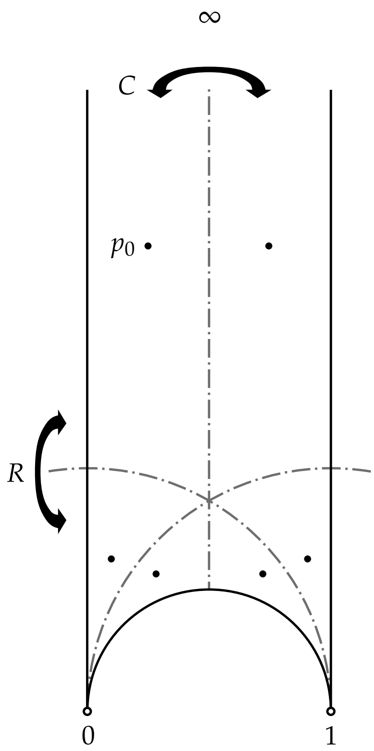


Figure 2.3.1: The ideal triangle $A_0 \subset \mathbb{H}^3$ with vertices $\{0, 1, \infty\}$, seen here in the upper-halfspace model. The Möbius transformations R and C act on A_0 as reflections through the altitudes emanating from 1 and ∞, respectively, and on the ambient $\mathbb{H}^3$ as 180-degree rotations about the said altitudes. The six dots represent the orbit of a single point p_0 lying in the interior of A_0 under the action of the group D_3 generated by R and C.

triangle $A_0 = T_{0,1,\infty}$ with vertices $\{0, 1, \infty\}$. Any orientation-preserving isometry of $\mathbb{H}^3$ leaving A_0 invariant must permute the ideal vertices of A_0, hence belong to the six-element dihedral group D_3 acting on A_0. Q.E.D.

Let $A_0 = T_{0,1,\infty}$ be the ideal triangle in $\mathbb{H}^3$ with vertices $\{0, 1, \infty\}$. The dihedral group of A_0 mentioned in the above proof, preserving the orientation of the ambient space $\mathbb{H}^3$, is generated by the Möbius transformations

$$C(z) = 1 - z \quad \text{and} \quad R(z) = \frac{1}{z}.$$

The map C (called 'complement') exchanges the vertices 0 and 1 while leaving ∞ invariant, whereas R (inversion) exchanges 0 and ∞ while fixing 1. Note that the Möbius transformations C and R satisfy the relations $C^2 = R^2 = (CR)^3 = \mathrm{Id}$. In this way, we recover the usual presentation of the dihedral group

$$D_3 \cong \langle C, R \mid C^2, R^2, (CR)^3 \rangle.$$

On a historical note, we remark that this six-element group of Möbius transformations was first described by Hathaway in [24].

Let $p_0 \in \mathring{A}_0$ be an arbitrarily fixed point in the interior of the triangle A_0 not lying on any of the altitudes of A_0. Then the orbit $D_3(p_0)$ consists of six points, as illustrated in Figure 2.3.1.

Although two different edges of the same face might be glued together in $\mathcal{T}$, the interiors of the faces of $\mathcal{T}$ are locally isometrically embedded open ideal triangles in M. Let $F \subset M$ be the interior of any face of $\mathcal{T}$. There exists an orientation-preserving geometric coordinate chart φ defined on an open neighbourhood of F in M which sends F to the open ideal triangle $\mathring{A}_0$. Furthermore, by Lemma 2.3.2, φ is unique up to composition with a symmetry of A_0. In other words, if φ' also maps F to $\mathring{A}_0$, $\varphi' = d \circ \varphi$ for some $d \in D_3$. Hence, the collection of six

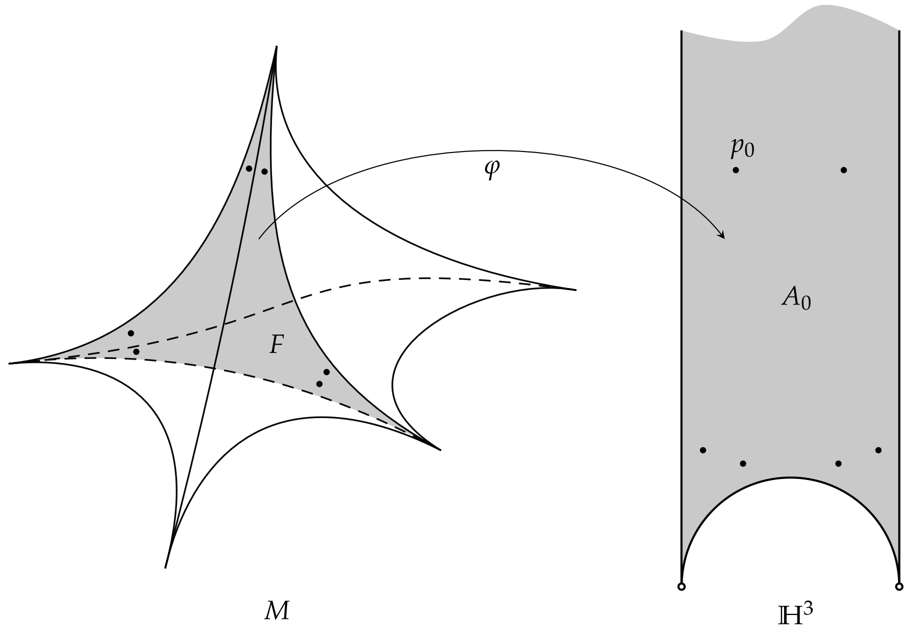


Figure 2.3.2: The interior F of a face of the ideal triangulation $\mathcal{T}$ of M can be mapped to the interior of A_0 by the means of an orientation-preserving geometric coordinate chart φ. This chart is unique up to a composition with a symmetry of A_0. Hence, the pre-image in M of the orbit $D_3(p_0)$ is a well-defined collection of six points inside F, called *special points on F*.

points $\varphi^{-1}(D_3(p_0)) \subset F$ does not depend on the choice of φ; see Figure 2.3.2.

Assumption 2.3.3. In the following, we assume that the point $p_0 \in \mathring{A}_0$ is fixed once and for all. Furthermore, we assume that p_0 lies in the top-left sextant of the triangle A_0, as shown in Figure 2.3.1. It suffices to assume that in the upper-halfspace model of $\mathbb{H}^3$, p_0 corresponds to a point $(z, t) \in \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3$ with $z \in (0, \frac{1}{2}) \subset \mathbb{R}$ and $t > 1$.

Definition 2.3.4. Let $\mathcal{T}$ be a geometric ideal triangulation of the manifold M with N tetrahedra. When F is the interior of a face of $\mathcal{T}$, we define $S_F \subset F$ by

$$S_F = \varphi^{-1}(D_3(p_0)),$$

where φ is any orientation-preserving geometric chart on an open neighbourhood of F in M satisfying $\varphi(F) = \mathring{A}_0$. We call elements of S_F *special points* on F. We also define

$$S_{\mathcal{T}} := \bigcup_{i=1}^{2N} S_{F_i},$$

where $F_1, \ldots, F_{2N}$ are the distinct open faces of $\mathcal{T}$.

The rigidity of ideal hyperbolic triangles implies that every special point $x \in S_F$ comes with a distinguished germ of an orientation-preserving developing map. Indeed, there is a unique germ $\varphi_x \in \mathscr{D}_x$ with the property that $\varphi_x(x) = p_0$ and the prolongation of φ_x takes the open triangle F locally isometrically onto $\mathring{A}_0$. The collection of these distinguished germs φ_x is a framing on $S_{\mathcal{T}}$ in the sense of Definition 2.2.5.

Definition 2.3.5. The framing $\{\varphi_x\}_{x \in S_{\mathcal{T}}}$ constructed above is called the *geometric framing* on $S_{\mathcal{T}}$.

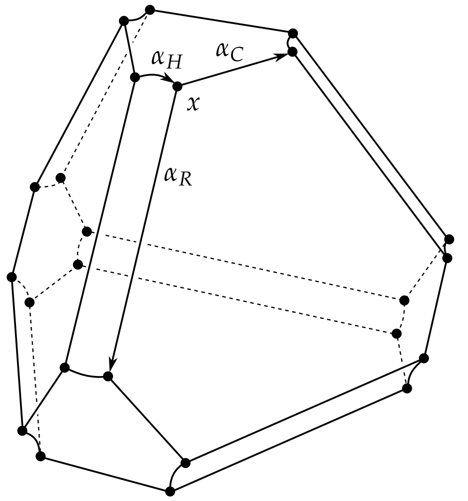


Figure 2.3.3: A tetrahedron of the triangulation $\mathcal{T}$ contains six objects of $\mathcal{G}(\mathcal{T})$ (basepoints) on each face. All morphisms in $\mathcal{G}(\mathcal{T})$ are generated by the homotopy classes of the paths shown in the figure. For every point $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$, we have paths α_R and α_C running from x to neighbouring basepoints on the same face. In addition, we consider paths running through the interior of the tetrahedron, connecting two basepoints lying vis-à-vis on adjacent faces. One such path is labeled α_H in the picture.

At present, we would like to use the geometric framings in order to describe the groupoid representation $\mathcal{H}ol$ in terms of paths connecting special points in M, thereby providing a geometric context to the construction mentioned in [12, Appendix B].

Definition 2.3.6. Let $\mathcal{T}$ be a geometric ideal triangulation of M. We define the groupoid $\mathcal{G}(\mathcal{T})$ to be the full subcategory of $\Pi_1(M)$ on the set of objects $S_{\mathcal{T}}$.

Remark 2.3.7. It is easy to see that $\mathcal{G}(\mathcal{T})$ is a groupoid. Indeed, since a full subcategory contains all morphisms of the original category, every morphism in $\mathcal{G}(\mathcal{T})$ still has an inverse.

We consider the following generating set of morphisms of $\mathcal{G}(\mathcal{T})$.

- For every $x \in S_F$, we take paths α_R and α_C contained in F and connecting x to the neighbouring points on F. The path α_R runs along an edge of F, whereas the path α_C remains close to an ideal vertex. These paths are depicted in Figure 2.3.4.
- For every edge e of a tetrahedron Δ, we take a pair of paths contained in Δ and connecting opposing basepoints on the two faces of Δ sharing the edge e. See Figure 2.3.3.

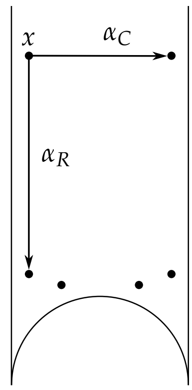


Figure 2.3.4: Some of the generating morphisms of $\mathcal{G}(\mathcal{T})$.

Note that in Figure 2.3.3 most edges are drawn undirected for the sake of clarity. The two ways of directing an edge represent two mutually inverse morphisms in $\mathcal{G}(\mathcal{T})$.

It is easy to see that this is a complete set of generators of $\mathcal{G}(\mathcal{T})$. Indeed, consider the graph G embedded in M and formed by gluing, for every tetrahedron, the graphs shown in Figure 2.3.3 according to the face identification pattern defining the triangulation $\mathcal{T}$. It is clear that any path in M connecting two points of $S_{\mathcal{T}}$ can be homotoped onto G, and thus is homotopic to a composition of certain directed edges of G.

2.3.2 Groupoid description of the holonomy representation

Since M is path-connected, the groupoids $\pi_1(M,*)$ and $\Pi_1(M)$ are equivalent in the sense of equivalence of categories. A small modification of the same reasoning shows that $\mathcal{G}(\mathcal{T})$ is equivalent to $\pi_1(M,*)$, hence also to $\Pi_1(M)$. Therefore, as observed in [17], in order to understand the groupoid representation $\mathcal{H}ol$, it suffices to understand its restriction to $\mathcal{G}(\mathcal{T})$. The main advantage of $\mathcal{G}(\mathcal{T})$ over $\Pi_1(M)$ is that $\mathcal{G}(\mathcal{T})$ is finitely generated. Thus, in order to describe $\mathcal{H}ol$, it suffices to understand the images of the generator paths α_R, α_C and α_H seen in Figure 2.3.3. We are going to use the concept of partial monodromy with respect to the geometric framing (Definition 2.3.5) to write down the images of the generator paths as elements of $PSL_2\mathbb{C}$.

The theorem stated below is not a new result. Rather than that, it constitutes an attempt at making the construction described in Appendix B to [12] completely rigorous and explaining its underlying geometric motivations. A somewhat similar method, albeit not using groupoids, is outlined in [44].

Theorem 2.3.8. *With respect to the geometric framing on $S_{\mathcal{T}}$,*

(i) the partial monodromy along any path α_R is the Möbius transformation $R = R^{-1}$;

(ii) the partial monodromy along any path α_C is the Möbius transformation $C = C^{-1}$.

(iii) Near an edge of a tetredron Δ, a path α_H travelling counter-clockwise around an ideal vertex of Δ, as in Figure 2.3.3, has partial monodromy H_s, where s is the shape parameter associated to the respective edge and the Möbius transformation H_s is given by

$$H_s(z) = sz.$$

The above theorem essentially says that in order to describe $\mathcal{H}ol$, we only need to consider the following Möbius transformations:

$$C(z) = 1 - z, \qquad R(z) = \frac{1}{z}, \qquad H_s(z) = sz, \tag{2.3.1}$$

where s is a shape parameter of a tetrahedron in the triangulation $\mathcal{T}$.

Proof. Let $x \in S_{\mathcal{T}}$ be any special point of the triangulation. Consider a path $\alpha_R \colon x \to y$ running along an edge of the face on which x lies. Denote by φ_x and φ_y the geometric framing germs at x and y, respectively. Then the continuation of φ_x along α_R is a germ $c_{\alpha_R,\varphi_x} \in \mathscr{D}_y$ satisfying $c_{\alpha_R,\varphi_x}(y) = R(p_0)$ (see Figure 2.3.1). On the other hand, by definition of geometric framing, $\varphi_y(y) = p_0$. Since both maps are onto the interior of the unit ideal triangle $A_0 \subset \mathbb{H}^3$,

$$c_{\alpha_R,\varphi_x} = R \circ \varphi_y.$$

Comparing to (2.2.1), we see that the partial monodromy along α_R equals $R^{-1} = R$. This proves Part (i) of the Theorem; Part (ii) is proved analogously.

For Part (iii), assume that F_x and F_y are faces, not necessarily distinct, of a tetrahedron Δ, meeting along an edge of Δ with shape parameter s. Let $x \in S_{F_x}(p_0)$ and $y \in S_{F_y}(p_0)$ be opposing special points located near the common edge of F_x and F_y in such a way that the path $\alpha_H \subset \Delta$ from x to y travels counter-clockwise around the ideal vertex (hence *clockwise* around the edge), as seen from the ideal vertex. This situation is illustrated in Figure 2.3.5. As before, let $\varphi_x \in \mathscr{D}_x$ and $\varphi_y \in \mathscr{D}_y$ be the geometric framing germs. By definition of φ_x, we have $\varphi_x(x) = p_0$ and $\varphi_x(\mathring{F}_x) = \mathring{T}_{0,1,\infty}$, where $T_{0,1,\infty} = A_0 \subset \mathbb{H}^3$ is the ideal triangle with vertices $\{0, 1, \infty\}$. Since φ_x is orientation-preserving, the continuation of φ_x along α_H must take F_y to the ideal triangle

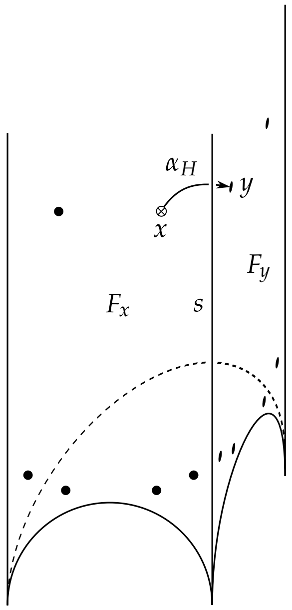


Figure 2.3.5: An illustration of the proof of Part (iii) of Theorem 2.3.8.

$T_{0,s^{-1},\infty}$. In particular, y is mapped to $H_{s^{-1}}(p_0)$, implying

$$c_{\alpha_H,\varphi_x} = H_s^{-1} \circ \varphi_y,$$

which shows that H_s is the partial monodromy along α_H. Q.E.D.

Using the groupoid $\mathcal{G}(\mathcal{T})$, we can easily write down a representative of the conjugacy class of the holonomy representation hol: $\pi_1(M,x) \to PSL_2\mathbb{C}$, where $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$. To this end, it suffices to homotope any loop α based at x onto the graph embedded in M whose intersection with each tetrahedron of $\mathcal{T}$ looks like Figure 2.3.3. This expresses the homotopy class of α as a composition of morphisms of $\mathcal{G}(\mathcal{T})$. Then $\mathrm{hol}([\alpha])$ can be deduced from the corresponding composition of partial monodromies; see Figure 2.3.6.

Note that when attempting to write down a representation of $\pi_1(M)$ using the monodromy of any locally constant sheaf, we need to be mindful of the necessity to take inverses in order to obtain a left representation; see equation (2.0.1). The holonomy representation constructed with the help of $\mathcal{G}(\mathcal{T})$ assigns to any $[\alpha] \in \pi_1(M,x)$ a composition of Möbius transformations C, R and H_s and their inverses. Since C and R are involutions, we do not need to pay attention to the orientations of the arrows within the faces of $\mathcal{T}$; the direction matters only when traversing the paths α_H which skip from one face to another. The relations corresponding to the faces of the polyhedron in Figure 2.3.3, verifiable directly from (2.3.1), assert that

$$\begin{aligned} H_s R H_s R &= \mathrm{Id} && \text{for every } s \in P = \mathbb{C}_{\mathrm{Im}>0}, \\ H_{z''} C H_{z'} C H_z C &= \mathrm{Id} && \text{whenever } z, z', z'' \text{ are related by (1.2.1)}, \\ (RC)^3 &= \mathrm{Id}. \end{aligned}$$

Although the tetrahedra in a geometric triangulation have shape parameters with positive imaginary parts, the above relations hold in fact for H_s with any $s \in \mathbb{C}_\times$. This is because the assignment $s \mapsto H_s$ is an analytic homomorphism $\mathbb{C}_\times \to PSL_2\mathbb{C}$.

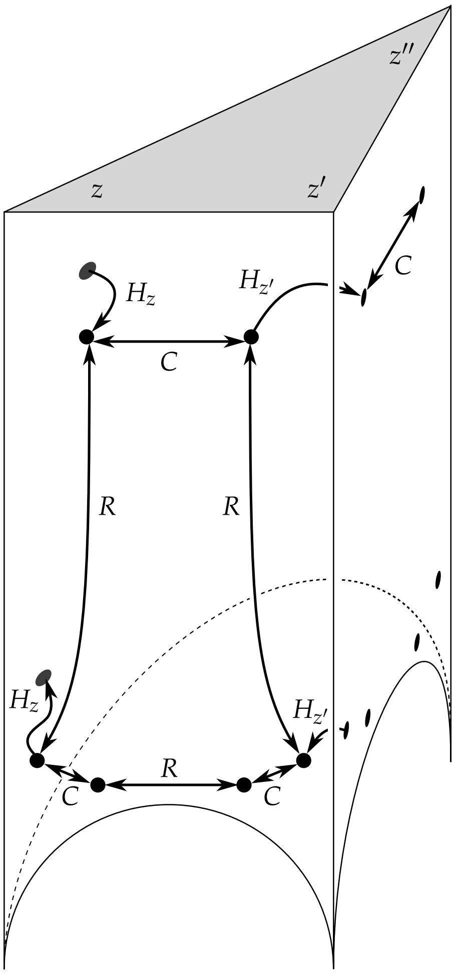


Figure 2.3.6: The image of the groupoid representation $\mathcal{H}ol$ restricted to $\mathcal{G}(\mathcal{T})$ can be written down in terms of partial monodromies of the generators with respect to the geometric framing. Since both C and R are of order two, we do not need to pay attention to directions of paths α_C and α_R. On the other hand, the homotheties H_s label paths traveling in the *counter-clockwise* direction when looking from ideal vertices. Here, s is the shape parameter of the corresponding edge. The inverses of these paths have partial monodromies $(H_s)^{-1} = H_{s^{-1}}$.

Example 2.3.9. We are going to calculate the holonomy image of the meridian μ of the figure-eight knot, shown in Figure 1.2.5. After representing the homotopy class of μ as a composition of morphisms of $\mathcal{G}(\mathcal{T})$, we obtain the thick curve of Figure 2.3.7, which we base at the lower side of the fundamental parallelogram. Since the figure shows a fundamental domain of the boundary torus as viewed from infinity, and μ appears to travel counterclockwise around the edge with parameter z_1'' and clockwise around the edge with parameter z_2'', the monodromy of $\mathscr{D}$ along μ is the product $H_{z_2''} C H_{z_1''}^{-1} C$. The image of $[\mu] \in \pi_1(M, *)$ under the holonomy representation can be then written as the inverse of the above element:

$$\operatorname{hol}([\mu]) = C H_{z_1''} C H_{z_2''}^{-1} = \left(z \mapsto \left(1 - \left(z_1'' \left(1 - \frac{1}{z_2''} z\right)\right)\right)\right) = \left(z \mapsto \frac{z_1''}{z_2''} z + (1 - z_1'')\right).$$

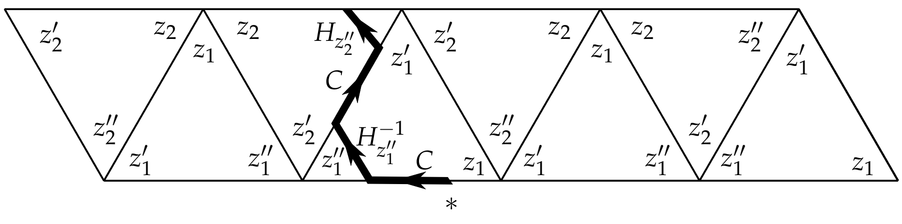


Figure 2.3.7: A calculation of the partial monodromy along the oriented curve μ of Figure 1.2.5, with the basepoint denoted $*$.

We observe that thanks to Assumption 2.3.3, the truncated ideal vertex is placed at $\infty \in \mathbb{C}P^1$ by the local chart φ_* (and in fact by all local charts of the geometric framing). As a consequence, the holonomy along μ is an affine map (cf. [40]). We verify that the linear part of $\mathrm{hol}([\mu])$ corresponds to the exponential of the left-hand side of the completeness equation (1.2.11).

Treating Möbius transformations as 2×2 matrices defined up to sign, we have

$$C = \pm \begin{bmatrix} i & -i \\ 0 & -i \end{bmatrix}, \qquad R = \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad H_s = \pm \begin{bmatrix} \sqrt{s} & 0 \\ 0 & \frac{1}{\sqrt{s}} \end{bmatrix}, \tag{2.3.2}$$

where $\sqrt{s}$ denotes the same branch of the square root in both entries of the matrix. Therefore, we immediately obtain the following corollary.

Corollary 2.3.10. *Suppose $\mathcal{T}$ is a geometric ideal triangulation of a 3-manifold M. There exists a representative of the conjugacy class of the holonomy representation* $\mathrm{hol}\colon \pi_1(M,*) \to PSL_2\mathbb{C}$ *which depends analytically on the shape parameters of the tetrahedra of the triangulation $\mathcal{T}$.*

2.3.3 Groupoid description of the adjoint representation

Suppose G is any group and $f\colon PSL_2\mathbb{C} \to G$ is a homomorphism. By applying the homomorphism f to the elements $C, R, H_s \in PSL_2\mathbb{C}$, we obtain immediately a groupoid description of the representation $f \circ \mathrm{hol}\colon \pi_1(M,x) \to G$ for any $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$. The most important case of this construction is the adjoint representation

$$\mathrm{Ad}\colon PSL_2\mathbb{C} \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C}),$$
$$\mathrm{Ad}(g)(v) = D_{\mathrm{Id}}(x \mapsto gxg^{-1})(v).$$

The remainder of this section is devoted to the study of the complex-linear representation

$$\mathrm{Ad} \circ \mathrm{hol}\colon \pi_1(M) \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C}), \tag{2.3.3}$$

which, just as hol itself, is defined only up to conjugation.

We wish to interpret the adjoint holonomy representation (2.3.3) geometrically. Recall that $\mathscr{K}$ denotes the sheaf of germs of Killing vector fields on M. By a theorem of Matsushima and Murakami, stated here as Part (ii) of Theorem 1.1.4, $\mathscr{K}$ is isomorphic to the local system determined by the representation $\mathrm{Ad} \circ \mathrm{hol}$. A geometric description of this relationship arises from the study of the analytic continuation of germs of Killing vector fields.

Recall that in (1.1.3) we had an isomorphism $\psi\colon \mathfrak{sl}_2\mathbb{C} \xrightarrow{\cong} \mathscr{K}(\mathbb{H}^3)$. We are going to construct its local analogues at the special points of M. For any $x \in S_{\mathcal{T}}$, denote by $\varphi_x \in \mathscr{D}_x$ the geometric framing germ at x. There exists a simply connected neighbourhood U of x onto which φ_x can be prolongated unambiguously, yielding an orientation-preserving geometric chart $\varphi\colon U \to \mathbb{H}^3$. Hence, we have a map

$$\psi_x\colon \mathfrak{sl}_2\mathbb{C} \xrightarrow{\cong} \mathscr{K}_x, \qquad \psi_x(v) = (\varphi_x^{-1})_*\psi(v), \qquad x \in S_{\mathcal{T}}. \tag{2.3.4}$$

Suppose $x, y \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$ and $\alpha\colon x \to y$ is a morphism of $\mathcal{G}(\mathcal{T})$. Consider the composition

$$G_\alpha\colon\ \mathfrak{sl}_2\mathbb{C} \xrightarrow{\psi_x} \mathscr{K}_x \xrightarrow{C_\alpha} \mathscr{K}_y \xrightarrow{\psi_y^{-1}} \mathfrak{sl}_2\mathbb{C},$$

where $C_\alpha\colon \mathscr{K}_x \to \mathscr{K}_y$ denotes analytic continuation of germs of Killing fields along α.

Proposition 2.3.11. *We have*

$$G_\alpha = \mathrm{Ad}(g_\alpha),$$

where g_α is the partial monodromy along α with respect to the geometric framing.

At the fundamental level, the above proposition is a corollary of Part (ii) of Theorem 1.1.4. Below, we show how one can prove this statement by using the restriction of the groupoid representation $\mathcal{H}o\ell$ to $\mathcal{G}(\mathcal{T})$.

Proof. Let $\alpha\colon x \to y$ be any morphism of $\mathcal{G}(\mathcal{T})$. The analytic continuation map $C_\alpha\colon \mathscr{K}_x \to \mathscr{K}_y$ can be written as

$$C_\alpha(X) = \left(c_{\alpha,\varphi_x}^{-1}\right)_* \left(E_{\mathbb{H}^3}\left((\varphi_x)_* X\right)\right),$$

where $E_{\mathbb{H}^3}\colon \mathscr{K}_{p_0} \to \mathscr{K}(\mathbb{H}^3)$ is the unique prolongation of a germ of a Killing vector field onto all of $\mathbb{H}^3$. Recall that the map $\psi\colon \mathfrak{sl}_2\mathbb{C} \xrightarrow{\cong} \mathscr{K}(\mathbb{H}^3)$ can be written as

$$\psi(v) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv)$$

where $\exp\colon \mathfrak{sl}_2\mathbb{C} \to PSL_2\mathbb{C}$ is the exponential map and $\exp(tv)$ is treated as a hyperbolic isometry, i.e., a map $\mathbb{H}^3 \to \mathbb{H}^3$. Let $v \in \mathfrak{sl}_2\mathbb{C}$ be arbitrary. We have

$$\begin{aligned}
G_\alpha(v) &= (\psi_y^{-1} \circ C_\alpha \circ \psi_x)(v) \\
&= \psi_y^{-1}\left(C_\alpha\left(\left.\frac{d}{dt}\right|_{t=0} \varphi_x^{-1}\left(\exp(tv)\right)_{p_0}\right)\right) \\
&= \psi_y^{-1}\left(\left.\frac{d}{dt}\right|_{t=0} \left(c_{\alpha,\varphi_x}^{-1} \circ E_{\mathbb{H}^3} \circ \exp(tv)_{p_0}\right)\right) \\
&= \psi^{-1} \circ E_{\mathbb{H}^3}\left(\left.\frac{d}{dt}\right|_{t=0} \varphi_y \circ c_{\alpha,\varphi_x}^{-1} \circ \exp(tv)_{g_\alpha^{-1}(p_0)}\right) && \text{— since } c_{\alpha,\varphi_x}(y) = g_\alpha^{-1}(p_0) \\
&= \psi^{-1} \circ E_{\mathbb{H}^3}\left(\left.\frac{d}{dt}\right|_{t=0} (g_\alpha \exp(tv) g_\alpha^{-1})_{p_0}\right) && \text{— by (2.2.1)} \\
&= \mathrm{Ad}\, g_\alpha(v). && \text{Q.E.D.}
\end{aligned}$$

Since for every $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$ we can identify the germ space $\mathscr{K}_x$ with the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ by the means of the map ψ_x of (2.3.4), we can compute the adjoint representation (2.3.3) purely in terms of the adjoint images of the Möbius transformations occurring in (2.3.2).

Thinking of elements of $PSL_2\mathbb{C}$ as of '$SL_2\mathbb{C}$ matrices defined up to sign', we can describe the adjoint action more explicitly by writing

$$\mathrm{Ad}(\pm M)(A) = MAM^{-1}, \tag{2.3.5}$$

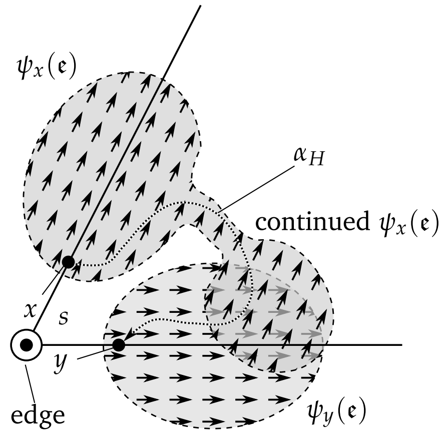


Figure 2.3.8: An angle of a horospherical triangle passing through the special points x, y in a tetrahedon Δ. The germ of a Killing vector field $\psi_x(\mathfrak{e}) \in \mathscr{K}_x$ acts as an infinitesimal translation of the horosphere and is shown on a neighbourhood of x. The analytic continuation of $\psi_x(\mathfrak{e})$ through the interior of Δ towards the face containing the point y can be compared with the Killing field $\psi_y(\mathfrak{e})$, which is depicted on a neighbourhood of y in the lower part of the figure. The identity $C_{\alpha_H}\psi_x(\mathfrak{e}) = s\psi_y(\mathfrak{e})$ can be directly observed on the overlap of the two neighbourhoods.

where $M \in SL_2\mathbb{C}$ and A is an arbitrary element of $\mathfrak{sl}_2\mathbb{C}$, i.e., a traceless 2×2 matrix with complex entries. Note that the right-hand side of (2.3.5) is well-defined, since $-\mathrm{Id}$ is central in the algebra $\mathcal{M}_{2\times 2}\mathbb{C}$. In order to write down elements of $\mathrm{Aut}(\mathfrak{sl}_2\mathbb{C})$ as 3×3 matrices, we shall use the ordered basis $(\mathfrak{e}, \mathfrak{h}, \mathfrak{f})$ defined in (1.1.4). With respect to this basis, we have

$$\mathrm{Ad}\left(\pm\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & -ab & -b^2 \\ -2ac & ad+bc & 2bd \\ -c^2 & cd & d^2 \end{bmatrix}. \tag{2.3.6}$$

Applying the above formula to (2.3.2), we obtain

$$\mathrm{Ad}\,C = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathrm{Ad}\,R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathrm{Ad}\,H_s = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/s \end{bmatrix}. \tag{2.3.7}$$

Hence, the adjoint representation $\mathrm{Ad} \circ \mathrm{hol}\colon \pi_1(M) \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C})$ can be constructed by mapping generators of the groupoid $\mathcal{G}(\mathcal{T})$ to the matrices listed above.

Example 2.3.12 (Continuing a germ through a dihedral angle)**.** From the form of the matrix $\mathrm{Ad}\,H_s$ in (2.3.7), we see that $\mathrm{Ad}\,H_s(\mathfrak{e}) = s\mathfrak{e}$. Recall that the Killing vector field $\psi(\mathfrak{e})$ is an infinitesimal horizontal translation of the upper halfspace model of $\mathbb{H}^3$, hence an infinitesimal translation of any horosphere in $\mathbb{H}^3$ centered at infinity, as shown in Figure 1.1.1.

Consider a horospherical triangle obtained by truncating an ideal vertex of an ideal tetrahedron Δ of shape parameter s. We can choose the truncation horosphere in such a way that it contains all special points on the faces of Δ that it intersects. Let x, y be special points lying vis-à-vis across an edge with shape parameter s, and let $\alpha_H\colon x \to y$ be a path winding clockwise around that edge when seen from the ideal vertex. Then α_H can be homotoped to lie within the truncating horosphere, resulting in the situation depicted in Figure 2.3.8, where α_H is represented by the dotted curve. We have

$$s\mathfrak{e} = \mathrm{Ad}\,H_s(\mathfrak{e}) = G_{\alpha_H}(\mathfrak{e}) = (\psi_y^{-1} \circ C_{\alpha_H} \circ \psi_x)(\mathfrak{e}),$$

which is equivalent to the equality $C_{\alpha_H}\psi_x(\mathfrak{e}) = s\,\psi_y(\mathfrak{e})$. We can "see" this equality in Figure 2.3.8. Map the horospherical triangle isometrically into $\mathbb{C}$, placing the intersection with the edge at the origin, and consider the holomorphic vector fields obtained by restricting $\psi_x(\mathfrak{e})$ and $\psi_y(\mathfrak{e})$. Then the holomorphic vector field $\psi_x(\mathfrak{e})$ is the direct image under the complex homothety $z \mapsto sz$ of the field $\psi_y(\mathfrak{e})$.

After looking again at Figure 2.3.2, we observe that every point in the orbit $D_3(p_0) \subset A_0 \subset \mathbb{H}^3$ lies 'near' to one of the edges of the ideal triangle A_0, with exactly two points 'near' every edge. Moreover, these two points correspond to the two ends of the edge. In order to make this notion of being 'near' precise, it suffices to observe that the action of D_3 permutes the sextants into which an ideal triangle is subdivided by its altitudes, and every sextant is adjacent to exactly one end of exactly one edge (see Figure 2.3.1). For example, the point p_0 in the upper-left sextant lies next to the edge $(0, \infty)$ near the end at ∞. Thus, another way of stating Assumption 2.3.3 is to require p_0 to lie next to the end at ∞ of the edge $(0, \infty)$ of the triangle A_0.

As a consequence of the above observations, every special point $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$ lies next to a unique edge of the triangulation $\mathcal{T}$ and near to one of the ends of that edge. This defines an assignment

$$\vec{e} : \mathrm{Ob}(\mathcal{G}(\mathcal{T})) \to \{\text{oriented edges of } \mathcal{T}\}$$

which takes any special point x to the unique edge of $\mathcal{T}$ near x. Further, the edge $\vec{e}(x)$ is oriented in the direction towards the end near to which x lies.

Proposition 2.3.13. *Let $\mathcal{T}$ be a geometric ideal triangulation of a 3-manifold M and let $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$. Then the map ψ_x of (2.3.4) satisfies*

$$\psi_x(\mathfrak{h}) = \mathfrak{h}_{\vec{e}(x)},$$

where $\mathfrak{h}_{\vec{e}(x)}$ is the Killing field defined by Proposition 1.1.5 on a connected, simply-connected neighbourhood of $\vec{e}(x)$ large enough to contain x.

Proof. For any $x \in \mathrm{Ob}(\mathcal{G}(\mathcal{T}))$, the geometric framing germ φ_x can be analytically continued onto $\vec{e}(x)$ along paths contained, except for their endpoints, in the open face of $\mathcal{T}$ containing x. By Assumption 2.3.3, this prolongation of φ_x takes $\vec{e}(x)$ to the oriented geodesic $\overrightarrow{(0, \infty)} \subset \mathbb{H}^3$. Since $\mathfrak{h} \in \mathscr{K}(\mathbb{H}^3)$ acts on the geodesic $(0, \infty)$ as a unit speed infinitesimal translation towards infinity, the pullback $\psi_x(\mathfrak{h}) \in \mathscr{K}_x$ is the germ at x of $\mathfrak{h}_{\vec{e}(x)}$. Q.E.D.

Remark 2.3.14. The representative of the conjugacy class of $\mathrm{Ad} \circ \mathrm{hol}$ constructed above depends rationally on the shape parameters of the ideal triangulation. Indeed, all matrices that result from multiplying together any combination of the matrices occurring in (2.3.7), as well as the inverses of these matrices, have entries that are rational functions of the shape parameters.

Chapter 3

Reidemeister torsion and its normalization

In this chapter, we define the notion of *combinatorial torsion* of a cochain complex over a field $\mathbb{F}$ and explain the construction of the non-abelian Reidemeister torsion invariant of cusped hyperbolic 3-manifolds, with values in $\mathbb{C}_\times/\{\pm 1\}$. This invariant was originally defined by J. Porti [46] in terms of a cellular chain complex of a 3-manifold M, twisted with the adjoint of the holonomy representation of the hyperbolic structure. Our definition is slightly simplified in comparison to the treatment in [46] and is stated in terms of twisted *co*chains.

Porti's definition of hyperbolic Reidemeister torsion requires the use of a so-called *geometric basis (base géométrique)* of the twisted cellular chain complex. We restate Porti's definition of a geometric basis in a coordinate-free way and explain how these geometric bases can be replaced with "Čech geometric bases" of Čech cochains associated to a good open cover or with "Steenrod geometric bases" of a cellular cochain complex with coefficients in a local system. We refer the reader to [20, 49, 14] for preliminary material on homology and cohomology with local coefficients.

In fact, geometric bases can be defined whenever the (co)chain complex is twisted by a unimodular representation of the fundamental group $\varrho\colon \pi_1(M) \to SL(n, \mathbb{F})$, though our primary interest is in $\mathbb{F} = \mathbb{C}$. The unimodularity assumption ensures that the top exterior power of the flat bundle $E = E_\varrho$ is trivial as a flat bundle. A geometric basis is then essentially a basis whose volume agrees with a prescribed global, constant section of $\bigwedge^n E_\varrho$ everywhere on M. These constant sections can be identified with elements of $\mathbb{F}$ once an isomorphism of the flat bundles $\det E_\varrho \cong \bigwedge^n E_\varrho$ is fixed.

Although the above interpretation of a geometric basis is not described in [46], it follows easily from basic K-theoretic considerations. Section 3.6 is highly technical and explains in detail how this interpretation can be applied to Čech cochain complexes. In Section 3.7, we do the same for cellular cochains using Steenrod's [49] definition, except we skip many of the details, which are analogous to the Čech case.

A reader familiar with Reidemeister torsion, but not with Porti's construction of the adjoint torsion, may wish to skip directly to Section 3.5, only referring back to earlier sections as needed. A reader completely unfamiliar with Reidemeister torsion is advised to start with Section 3.1.

3.1 Definition and basic properties of combinatorial torsion

In this section, we review the definition of the combinatorial torsion of a finite-dimensional cochain complex over a field. Our treatment is based on the book by V. Turaev [53] and the article by J. Milnor et al. [36].

Suppose that $\mathbb{F}$ is a field and consider a finite cochain complex over $\mathbb{F}$

$$C^\bullet = (0 \to C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-2}} C^{d-1} \xrightarrow{\delta^{d-1}} C^d \to 0),$$

in which all cochain groups are finite-dimensional $\mathbb{F}$-vector spaces, equipped with fixed bases $\underline{c^i} \subset C^i$. Given any other basis $\underline{a^i} \subset C^i$, we denote by $\det[\underline{a^i}/\underline{c^i}]$ the determinant of the change-of-basis matrix whose columns are the coefficients of vectors of $\underline{a^i}$ written in the basis $\underline{c^i}$. In other words,

$$\det\left[\underline{a^i}/\underline{c^i}\right] = \pm \det M_i, \text{ where } M_i\colon C^i \to C^i \text{ satisfies } M_i(\underline{c^i}) = \underline{a^i}.$$

This quantity is well-defined only up to sign, since the bases in question are not ordered.

We shall use the standard notation $B^i = \mathrm{Im}(\delta^{i-1}) \subset C^i$, $Z^i = \mathrm{Ker}(\delta^i) \subset C^i$; B^i is called the space of i-coboundaries, whereas Z^i is the space of i-cocycles. For every i, we have a short exact sequence

$$0 \to Z^i \xrightarrow{\subseteq} C^i \xrightarrow{\delta^i} B^{i+1} \to 0.$$

Since we work over a field, this sequence always has a splitting $s^i\colon B^{i+1} \to C^i$. Denote by $H^i = Z^i/B^i$ the ith cohomology group of $C^\bullet$ and by $\underline{h^i}$ a preferred basis of H^i. We write $\underline{c} = \sqcup_i \underline{c^i}$ and $\underline{h} = \sqcup_i \underline{h^i}$ for short. Note that given a collection of bases $\underline{b^i}$ of B^i where $i \in \{0, \ldots, d\}$, we can form a new basis of C^i defined as $\underline{b^i} \sqcup \underline{\widetilde{h^i}} \sqcup s^i(\underline{b^{i+1}})$, where $\underline{\widetilde{h^i}}$ consists of cocycles representing the cohomology classes of the elements of $\underline{h^i}$. The Reidemeister torsion of $C^\bullet$ is then defined as

$$\mathbb{T}(C^\bullet, \underline{c}, \underline{h}) = \pm \prod_{i=0}^{d} \det\left[\underline{b^i} \sqcup \underline{\widetilde{h^i}} \sqcup s^i(\underline{b^{i+1}})/\underline{c^i}\right]^{(-1)^i} \in \mathbb{F}_\times/\{\pm 1\}, \tag{3.1.1}$$

and only depends on the choice of the bases $\underline{c}$ and $\underline{h}$. In particular, the torsion depends neither on the choice of the bases $\underline{b^i}$ nor on the choice of the sections s^i. We refer to the first chapter of [53] for more details and proofs.

Remark 3.1.1. The exponent $(-1)^i$ used in (3.1.1) could be replaced by $(-1)^{i+1}$, which would have the effect of replacing $\mathbb{T}$ with $1/\mathbb{T}$. Another way to think about this choice is that if the grading of the cochain complex is shifted by one in either direction, we obtain $\mathbb{T}(C^{\bullet\pm 1}) = 1/\mathbb{T}(C^\bullet)$. In general, there is no "natural" way to choose which of the two numbers to call torsion. Jérôme Dubois [13] interpreted Reidemeister torsion as a volume form on the $SU(2)$ character variety of a knot group, which requires the torsion to depend multilinearly on a basis of the first cohomology groups H^1 (naturally identified with the fibres of the tangent bundle to the character variety at smooth points). In order for this construction to work as intended, Dubois had to set the exponent to $(-1)^{i+1}$; see also [18].

Example 3.1.2 (Remark 1.4-3 in [53])**.** Consider the cochain complex

$$0 \to C^1 \xrightarrow{\delta^1} C^2 \to 0$$

with fixed bases $\underline{c^1} \subset C^1$ and $\underline{c^2} \subset C^2$. Assume furtermore that δ^1 is an isomorphism. Then $H^1 = Z^1/B^1 = \operatorname{Ker}\delta^1/0 = 0$ and $H^2 = Z^2/B^2 = \operatorname{Im}\delta^2/C^2 = 0$. Take $s^1 = (\delta^1)^{-1}$, $\underline{b^1} = \varnothing$, $\underline{b^2} = \underline{c^2}$. We obtain

$$\mathbb{T}(C^\bullet, \underline{c^1} \sqcup \underline{c^2}, \varnothing) = \pm \det\left[(\delta^1)^{-1}(\underline{c^2})/\underline{c^1}\right]^{-1} \det\left[\underline{c^2}/\underline{c^2}\right].$$

The second factor of the above product equals 1 and the first factor is the reciprocal of the determinant of a matrix expressing images of the vectors of $\underline{c^2}$ under $(\delta^1)^{-1}$ in terms of the basis $\underline{c^1}$. Hence, $\mathbb{T}(C^\bullet, \underline{c^1} \sqcup \underline{c^2}, \varnothing)$ is equal, up to sign, to the determinant of the matrix of δ^1 with respect to the bases $\underline{c^1}$, $\underline{c^2}$.

The above example shows that the notion of combinatorial torsion generalizes the concept of the determinant of a linear isomorphism with respect to prescribed bases of its domain and codomain. In a sense, the torsion is a systematic way to take the 'determinant of a cochain complex' with respect to fixed bases of the cochain spaces and the cohomology groups. The sign indeterminacy of torsion stems from the fact that we are not specyfing any particular ordering of the bases used.

Remark 3.1.3. Vladimir Turaev invented a method of removing the sign indeterminacy of Reidemeister torsion by fixing a *homology orientation* (which, in our language, should perhaps be called a *cohomology orientation*). The details of this construction can be found in Chapter III of Turaev's book [53]. We do not study sign-refined torsions in this work.

The combinatorial torsion depends on the chosen bases of the cochain spaces and the cohomology groups in a predictable way. The following theorem can be found e.g. in [46].

Theorem 3.1.4 (Change of basis formula)**.** *Suppose $\underline{c}, \underline{\hat{c}}$ are two bases of the cochain complex $C^\bullet$ and $\underline{h}, \underline{\hat{h}}$ are two bases of its cohomology module $H^\bullet$. Then*

$$\mathbb{T}(C^\bullet, \underline{\hat{c}}, \underline{\hat{h}}) = \mathbb{T}(C^\bullet, \underline{c}, \underline{h}) \prod_{i=0}^{d} \left(\det\left[\underline{c^i}/\underline{\hat{c}^i}\right] \det\left[\underline{\hat{h}^i}/\underline{h^i}\right]\right)^{(-1)^i}.$$

The proof of the above theorem is elementary and follows from the equality $\det[x/y]\det[y/z] = \det[x/z]$ which holds whenever x, y, z are three bases of the same vector space.

Combinatorial torsion is multiplicative with respect to short exact sequences of cochain complexes. Consider a short exact sequence of complexes

$$0 \to C_I^\bullet \xrightarrow{\iota^\bullet} C_{II}^\bullet \xrightarrow{\pi^\bullet} C_{III}^\bullet \to 0 \tag{3.1.2}$$

and equip these complexes with bases $\underline{c_I}$, $\underline{c_{II}}$ and $\underline{c_{III}}$, respectively. Also choose bases $\underline{h_I}$, $\underline{h_{II}}$ and $\underline{h_{III}}$ of the corresponding cohomology groups. We say that the bases $\underline{c_I}$, $\underline{c_{II}}$ and $\underline{c_{III}}$ are *compatible* if for every i, $\det\left[\iota^i(\underline{c_I^i}) \sqcup \sigma^i(\underline{c_{III}^i})/\underline{c_{II}^i}\right] = \pm 1$, where σ^i is a section of π^i, i.e., $\pi^i \circ \sigma^i = \mathrm{Id}$. This notion of compatibility was introduced by John Milnor in [36]. Let $\mathcal{H}^\bullet$ be the long exact sequence in cohomology associated to (3.1.2). Milnor proved the following useful identity.

Theorem 3.1.5 (cf. [36, Theorem 3.2])**.** *If the bases $\underline{c_I}$, $\underline{c_{II}}$ and $\underline{c_{III}}$ are compatible, then*

$$\mathbb{T}(C_{II}^\bullet, \underline{c_{II}}, \underline{h_{II}}) = \mathbb{T}(C_I^\bullet, \underline{c_I}, \underline{h_I})\, \mathbb{T}(C_{III}^\bullet, \underline{c_{III}}, \underline{h_{III}})\, \mathbb{T}(\mathcal{H}^\blacksquare, \underline{h_I} \sqcup \underline{h_{II}} \sqcup \underline{h_{III}}, \varnothing) \tag{3.1.3}$$

for a suitably chosen grading $\blacksquare$ of the complex $\mathcal{H}$.

We now wish to explain how to choose the grading ■ so that (3.1.3) holds as stated. The long exact sequence $\mathcal{H}$ can be written as

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(C_I^\bullet) & \longrightarrow & H^0(C_{II}^\bullet) & \longrightarrow & H^0(C_{III}^\bullet) & \\
& H^1(C_I^\bullet) & \longrightarrow & H^1(C_{II}^\bullet) & \longrightarrow & H^1(C_{III}^\bullet) & \\
& & & \cdots & & & \\
& H^d(C_I^\bullet) & \longrightarrow & H^d(C_{II}^\bullet) & \longrightarrow & H^d(C_{III}^\bullet) & \longrightarrow 0.
\end{array}$$

We see from (3.1.1) that for even i, $\mathbb{T}(C_{II}^\bullet, \underline{c_{II}}, \underline{h_{II}})$ depends multilinearly on $\underline{h_{II}^i}$. The requirement that the same be true of the right-hand side of (3.1.3) determines the parity of the correct grading ■ of $\mathcal{H}$. Namely, for i even, the terms $H^i(C_{II}^\bullet)$ must have *odd* degrees, since $\underline{h_{II}^i}$ is a basis of the space $H^i(C_{II}^\bullet)$, so it plays the role of the basis $\underline{c^i}$ in (3.1.1). Therefore, we must set

$$■: \qquad \mathcal{H}^{3n} = H^n(C_I^\bullet), \quad \mathcal{H}^{3n+1} = H^n(C_{II}^\bullet), \quad \mathcal{H}^{3n+2} = H^n(C_{III}^\bullet) \tag{3.1.4}$$

for all $n \in \mathbb{Z}$. See also the footnote on p. 367 of [36].

3.2 How to turn the torsion into a topological invariant

At present, we are interested in the potential of using the combinatorial torsion $\mathbb{T}$ defined in the preceding section to define topological invariants. It turns out that most of the time, the resulting invariants are only combinatorial invariants, i.e., they depend non-trivially on the exact way in which the topological space is represented as a CW-complex. While the combinatorial torsion is in general invariant under cell subdivision, the falsity of the Hauptvermutung implies that combinatorial invariance is insufficient to ensure topological invariance. However, there are special situations, usually in low dimension, where the torsion can still furnish topological invariants. This section contains an outline of the main ideas behind these special constructions, and subsequent sections fill in the big picture with details.

Given a finite, connected cell complex X and a complex linear representation ϱ of the fundamental group of X,

$$\varrho\colon \pi_1(X,*) \to GL(n,\mathbb{C}),$$

we can treat $\mathbb{C}^n$ as a left module over the group algebra $\mathbb{C}[\pi_1(X,*)]$. We are going to define the cellular cochain complex with twisted coefficients given by ϱ, which we shall denote $C^\bullet(X,\varrho)$. The *twisted Reidemeister torsion* of X with coefficients in ϱ is then the quantity

$$\mathbb{T}(C^\bullet(X,\varrho), \underline{c}, \underline{h}) \in \mathbb{C}_\times/\{\pm 1\}, \tag{3.2.1}$$

where $\underline{c}$ is a basis of the cochain complex $C^\bullet(X,\varrho)$ and $\underline{h}$ is a basis of its cohomology.

In general, the Reidemeister torsion (3.2.1) is *not* a topological invariant of the space X. Instead, it depends on the following data:

1. The topology of X,
2. The CW structure of X, or more specifically, the *simple homotopy type* of X; cf. [36],

3. The chosen basepoint $* \in X$,

4. The linear representation ϱ,

5. The basis $\underline{c}$ of $C^\bullet(X, \varrho)$,

6. The basis $\underline{h}$ of $H^\bullet(X, \varrho)$.

Therefore, in order for Reidemeister torsion to define a topological invariant of X, all of the additional choices listed in points 2–6 must be made in a canonical way.

The Reidemeister torsion of a cellular chain or cochain complex is invariant under cell subdivision [36, Theorem 7.1]. However, as mentioned above, this does not ensure the invariance of torsion even for manifolds, since the Hauptvermutung is false in high dimension [35]. In the following, we shall assume that X is a cell decomposition of a manifold of dimension at most 3, since in this case the topological invariance follows from [37, Theorem 4]. In particular, when X is a 3-manifold, the torsion is independent of the choice of the simplicial or cellular complex used to represent X.

We now discuss the problem of choosing a basis $\underline{c}$ of the cochain complex $C^\bullet(X, \varrho)$. When ϱ is a *unimodular* representation, there exists a preferred class of bases, which we shall call *geometric bases*, following Porti [46]. Recall that Theorem 3.1.4 describes the dependence of torsion on the the basis $\underline{c}$. It turns out that all *geometric bases* give the same value of $\mathbb{T}$ if the Euler characteristic of X is zero. This is a serendipity, since closed 3-manifolds and compact 3-manifods with toroidal boundary are known to have Euler characteristic zero.

Moreover, the torsion of the twisted cochain complex $C^\bullet(X, \varrho)$ only depends on the conjugacy class of the unimodular representation ϱ. Although conjugate representations always yield isomorphic local systems, we also need to know that the conjugation action preserves the class of geometric bases, so that the torsion only depends on the local system defined by ϱ [46, Rem. a2, p. 10]. As a consequence of the conjugacy invariance, the dependence of torsion on the choice of the basepoint $* \in X$ vanishes as well.

To summarize, when X is a connected 3-manifold of Euler characteristic zero, ϱ is unimodular and $\underline{c}$ is any geometric basis, the combinatorial torsion of (3.2.1) only depends on

1. The topology of X,

2. The conjugacy class of the unimodular representation ϱ,

3. The basis $\underline{h}$ of $H^\bullet(X, \varrho)$.

Note that a conjugacy class of a linear representation of ϱ of $\pi_1(X)$ can be thought of as a local system on X or as the corresponding flat vector bundle (see Chapter 2). From our perspective, the most important example of the representation ϱ will be the adjoint of the holonomy representation $\mathrm{Ad} \circ \mathrm{hol}$ discussed in Section 2.3.3. Note that this adjoint representation is unimodular (Example 3.3.6).

Joan Porti [46] constructed a Reidemeister torsion invariant defined for a complete hyperbolic 3-manifold M of finite volume with cusps equipped with a collection of homologically nontrivial peripheral curves, one at each cusp. The assumption of hyperbolicity ensures that, by Mostow-Prasad Rigidity and Remark 2.1.1, the representation $\varrho = \mathrm{Ad} \circ \mathrm{hol}$ is unique up to conjugation. This takes care of Point 2. in the above list. Note that complete hyperbolic manifolds are Eilenberg-MacLance spaces, since their universal covering spaces can be identified with $\mathbb{H}^3$.

Therefore, the homotopy type of a hyperbolic manifold is fully determined by its fundamental group or, equivalently, by its image under the faithful representation $\varrho = \mathrm{Ad} \circ \mathrm{hol}$.

Unfortunately, the local system defined by the adjoint holonomy representation has non-vanishing cohomology, so we are still left with the problem of fixing a basis $\underline{h}$. The essence of Porti's construction [46] is an elegant solution of this problem. More precisely, the choice of a suitable peripheral multicurve γ allows us to 'balance' the bases for the first and second cohomology groups in such a way that the torsion is a well-defined topological invariant of the ordered pair (M, γ).

In the following sections, we explain the main steps of the construction outlined above. We refer to Porti's monograph [46] for more details.

3.3 Twisted cochains and geometric bases

In this section, we discuss the problem of choosing the basis of a twisted cochain complex in a canonical way. Such choice is necessary in order for Reidemeister torsion to define a topological invariant.

We start by explaining how to normalize bases of the cochain spaces associated to a CW-complex and a representation of its fundamental group. The basic idea behind this construction was outlined by Milnor [36] and later developed into the concept of a *geometric basis (base géométrique)* by J. Porti [46], originally for $n = 3$.

We start by defining the well-known [49] notion of a cellular cochain complex with twisted coefficients. Let X be a finite path-connected CW-complex. Choose a basepoint $* \in X$. The universal covering space $\widetilde{X}$ inherits a CW-structure from X which is preserved by the action of $\pi_1(X, *)$ on $\widetilde{X}$. Given a representation $\varrho\colon \pi_1(X, *) \to GL(n, \mathbb{C})$, we define the cellular cochain complex of X twisted by ϱ as follows.

Definition 3.3.1. The *cellular cochain complex* $C^\bullet(X, \varrho)$ *of* X *twisted by* ϱ is given by

$$C^k(X, \varrho) = \{c\colon C_k(\widetilde{X}, \mathbb{Z}) \to \mathbb{C}^n \mid c\,(\gamma(\alpha)) = \varrho(\gamma)\,(c(\alpha)) \text{ for every } \gamma \in \pi_1(X, *)\}$$

along with the coboundary operators $\delta^k\colon C^k(X, \varrho) \to C^{k+1}(X, \varrho)$ defined by

$$\left[\delta^k(c)\right](\alpha) = c(\partial\alpha)$$

for any chain $\alpha \in C_{k+1}(\widetilde{X}, \mathbb{Z})$ and any cochain $c \in C^k(X, \varrho)$.

In other words, $C^k(X, \varrho)$ consists of $\pi_1(X, *)$-equivariant maps from $C_k(\widetilde{X}, \mathbb{Z})$ into $\mathbb{C}^n$, where the action of $\pi_1(X, *)$ on $C_k(\widetilde{X}, \mathbb{Z})$ is induced by its action on $\widetilde{X}$ via deck transformations.

We now explain how one can endow each space $C^k(X, \varrho)$ with an explicit basis. This construction leads to the concept of a *geometric basis* and is dual to the construction in [46].

Let $\underline{b} = \{b_1, \ldots, b_n\}$ be a fixed basis of $\mathbb{C}^n$. The spaces $C^k(X, \varrho)$ are finite-dimensional and can be understood in terms of a choice of lifts of k-cells of X to the universal covering complex $\widetilde{X}$. Consider the finite set $S^{(k)}$ of all k-dimensional cells of X. For each k-cell $\Delta \in S^{(k)}$, choose a lift $\widetilde{\Delta}$ of Δ to $\widetilde{X}$. For $1 \leq i \leq n$, define the cochains $c_\Delta^{(i)} \in C^k(X, \varrho)$ by

$$c_\Delta^{(i)}(\widetilde{\Delta}) = b_i \text{ and } c_\Delta^{(i)}(\Gamma) = 0 \text{ if } \Gamma \text{ is not a lift of } \Delta. \tag{3.3.1}$$

For a fixed $\Delta \in S^{(k)}$, the cochains $\{c_\Delta^{(i)}\}_{i=1}^n$ are linearly independent and span a vector subspace $V(\Delta) \subseteq C^k(X,\varrho)$. Furthermore, we have

$$C^k(X,\varrho) = \prod_{\Delta \in S^{(k)}} V(\Delta). \tag{3.3.2}$$

We define $\underline{c^k}$ by

$$\underline{c^k} = \bigcup_{\Delta \in S^{(k)}} \bigcup_{i=1}^{n} \left\{ c_\Delta^{(i)} \right\}.$$

The set $\underline{c^k}$ is a basis for the space of k-cochains $C^k(X,\varrho)$. By repeating the above procedure for every k, we obtain a basis $\underline{c} := \bigcup_k \underline{c^k}$ for the space $C^\bullet(X,\varrho) = \prod_k C^k(X,\varrho)$.

Definition 3.3.2 (Geometric basis – base géométrique)**.** Any basis $\underline{c}$ of $C^\bullet(X,\varrho)$ constructed as above is called a *geometric basis*.

Remark 3.3.3. We stress that there are two arbitrary choices involved in the construction of a geometric basis:

1. the choice of the fixed basis $\underline{b} \subseteq \mathbb{C}^n$;
2. the choice of a lift of each cell of X to $\widetilde{X}$.

Remark 3.3.4. Equivalently, $C^\bullet(X,\varrho)$ can be defined as a tensor product of bimodules. This alternative definition is used, for instance, in [18]. If $\mathcal{R} := \mathbb{Z}[\pi_1(X)]$, the abelian group $C^\bullet(\widetilde{X},\mathbb{Z})$ can be given the structure of a right $\mathcal{R}$-module under the action by deck transformations, whereas $\mathbb{C}^n$ is a left $\mathcal{R}$-module via the representation ϱ. Then $C^\bullet(X,\varrho) \cong C^\bullet(\widetilde{X},\mathbb{Z}) \otimes_{\mathcal{R}} \mathbb{C}^n$. Contemplating (3.3.1), we see that $\underline{c}$ consists of all tensor products of elements of $\underline{b}$ with elements of an $\mathcal{R}$-basis of the free $\mathcal{R}$-module $C^\bullet(\widetilde{X},\mathbb{Z})$ given by a choice of a lift of each cell to the universal covering space.

Definition 3.3.5. A linear representation $\varrho\colon G \to GL(n,\mathbb{C})$ of a group G is called *unimodular* if its image is contained in $SL(n,\mathbb{C})$.

Example 3.3.6. Consider the adjoint representation $\mathrm{Ad}\colon PSL_2\mathbb{C} \to \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C})$. We can identify $\mathfrak{sl}_2\mathbb{C}$ with $\mathbb{C}^3$ using the ordered basis $(\mathfrak{e},\mathfrak{h},\mathfrak{f})$ defined in (1.1.4). Then the image of Ad consists of matrices of the form given in (2.3.6). An elementary computation shows that these matrices have unit determinant. Hence, Ad is unimodular.

Remark 3.3.3 notwithstanding, it turns out that under suitable assumptions on X and ϱ, the torsion is constant on the class of geometric bases. Note that the bases we work with are unordered, so we treat the torsion as a non-zero scalar defined up to sign.

Proposition 3.3.7 (cf. [46, p. 10])**.** *Let X be a finite path-connected CW-complex with Euler characteristic zero and let ϱ be a unimodular representation of its fundamental group. Consider the twisted cellular cochain complex $C^\bullet(X,\varrho)$ and choose a basis $\underline{h}$ for its cohomology $H^\bullet(X,\varrho)$. If $\underline{c}$ and $\underline{c'}$ are two geometric bases of $C^\bullet(X,\varrho)$, then*

$$\mathbb{T}(C^\bullet(X,\varrho),\underline{c},\underline{h}) = \mathbb{T}(C^\bullet(X,\varrho),\underline{c'},\underline{h}).$$

Proof. According to Theorem 3.1.4, the quotient of the two torsions equals

$$Q := \pm \prod_{k=0}^{d} \det\left[\underline{c^k}/\underline{c'^k}\right]^{(-1)^k},$$

where d is the degree of the CW-complex X. When $\underline{c} = \underline{c}'$, we trivially have $Q = 1$. Thus, we only need to observe the effect on Q of the two choices mentioned in Remark 3.3.3.

Firstly, suppose that $\underline{c}$ and $\underline{c}'$ are defined using the same lifts of all cells but two different fixed bases $\underline{b}$ and $\underline{b}'$, respectively. Then $\det\left[\underline{c^k}/\underline{c'^k}\right] = (\det[\underline{b}/\underline{b}'])^{a_k}$ where $a_k = \#S^{(k)} \in \mathbb{N}$ is the number of k-cells in X. Thus

$$Q = \pm\left(\det[\underline{b}/\underline{b}']\right)^{\sum_k(-1)^k a_k} = \pm 1,$$

since $\sum_k (-1)^k a_k = \chi(X) = 0$ by assumption.

Secondly, suppose that $\underline{c^k}$ and $\underline{c'^k}$ agree on all factors of (3.3.2) except for a factor $V(\Delta)$ corresponding to a k-cell Δ. Denote by $\widetilde{\Delta}$ the lift of Δ which was used to define $\{c_\Delta^{(i)}\}_{i=1}^n$ and by $\gamma(\widetilde{\Delta})$ the lift used in the definition of $\{c'^{(i)}_\Delta\}_{i=1}^n$, with $\gamma \in \pi_1(X) = \operatorname{Aut}\left(\widetilde{X} \searrow X\right)$. It follows from unimodularity of ϱ that $\det\left[\underline{c^k}/\underline{c'^k}\right] = \det\left[\{c_\Delta^{(i)}\}_{i=1}^n / \{c'^{(i)}_\Delta\}_{i=1}^n\right] = \det \varrho(\gamma) = 1$, implying $Q = \pm 1$. Q.E.D.

The above proposition allows us to view the Reidemeister torsion of $C^\bullet(X, \varrho)$ as an invariant of the triple $(X, \varrho, \underline{h})$, with the understanding that we always use a geometric basis for the cochain spaces. Since only the conjugacy class of ϱ affects the value of the Reidemeister torsion [46, Rem. a2, p. 10], it also makes sense to regard the combinatorial torsion as an invariant of the triple $(X, E, \underline{h})$, where $E = E_\varrho$ is the flat bundle determined by ϱ.

3.4 Cohomology of the sheaf of germs of Killing vector fields

Suppose M is a connected, complete hyperbolic 3-manifold of finite volume with $k > 0$ cusps. It follows that M has the homotopy type of a compact manifold $\bar{M}$ with the boundary $\partial\bar{M}$ consisting of k tori. As in Section 1.3, we fix the numbering of these tori and write

$$\partial\bar{M} = T_1 \cup \cdots \cup T_k.$$

The complete hyperbolic structure on M defines the holonomy representation $\mathrm{hol}\colon \pi_1(M) \to PSL_2\mathbb{C}$ and its adjoint $\mathrm{Ad}\circ\mathrm{hol}$ uniquely up to conjugation. For any cell decomposition of M, we can therefore consider the cellular cochain complex $C^\bullet(M; \mathrm{Ad}\circ\mathrm{hol})$ of Definition 3.3.1.

Recall that the symbol $\mathscr{K}$ denotes the sheaf of germs of Killing vector fields on M. By Theorem 1.1.4, Part (ii), the sheaf $\mathscr{K}$ is isomorphic to the local system defined by the representation $\mathrm{Ad}\circ\mathrm{hol}$, so the cohomology groups of the complex $C^\bullet(M; \mathrm{Ad}\circ\mathrm{hol})$ are isomorphic to the cohomology groups of the sheaf $\mathscr{K}$. We are now going to summarize certain facts about the cohomology of $\mathscr{K}$ which will be useful later.

Lemma 3.4.1 (Cohomology of $\mathscr{K}$)**.** *If M has k cusps, then*

$$H^0(M;\mathscr{K}) = 0, \quad H^1(M;\mathscr{K}) \cong \mathbb{C}^k, \quad H^2(M;\mathscr{K}) \cong \mathbb{C}^k, \quad H^d(M;\mathscr{K}) = 0 \text{ for } d > 2.$$

Proof. The cohomology group $H^0(M;\mathscr{K})$ is simply the space of global sections of $\mathscr{K}$. As explained in Section 1.1, $\mathscr{K}(M)$ is isomorphic to the Lie algebra of the group $\mathrm{Isom}^+(M)$ of orientation-preserving isometries of M – see in particular (1.1.2). Since M is a complete finite-volume hyperbolic 3-manifold, $\mathrm{Isom}^+(M)$ is finite and its Lie algebra trivial, implying $H^0(M;\mathscr{K}) = 0$.

The isomorphism $H^1(M;\mathscr{K}) \cong \mathbb{C}^k$ is a conseqeunce of Remark (ii) on p. 71 of [46].

Recall that the manifold M has the homotopy type of a compact manifold $\bar{M}$ with boundary. Suppose that $\bar{M}$ is represented as a simplicial complex with $\partial\bar{M}$ a subcomplex. By collapsing 3-simplices starting from the boundary, we see that $\bar{M}$ has the homotopy type of a 2-complex, called the *spine* of M. Hence, all cohomology groups of M in degree 3 and higher will vanish.

Finally, since $0 = \chi(M) = 3(\dim H^2(M;\mathscr{K}) - k)$, we must also have $H^2(M;\mathscr{K}) \cong \mathbb{C}^k$. Q.E.D.

By an abuse of notation, denote by $\partial\bar{M}$ a closed neighbourhood of the ends of M, consisting for instance of disjoint closed horoball neighbourhoods of the cusps. Since $\partial\bar{M}$ is closed in M, we can consider the sheaf $\mathscr{K}_{\partial\bar{M}}$ obtained by restriction. The restriction map $\mathscr{K} \to \mathscr{K}_{\partial\bar{M}}$ induces the linear maps

$$H^1(M;\mathscr{K}) \xrightarrow{r^1} H^1(M;\mathscr{K}_{\partial\bar{M}}) \quad \text{and} \quad H^2(M;\mathscr{K}) \xrightarrow{r^2} H^2(M;\mathscr{K}_{\partial\bar{M}}).$$

Equivalently, the above maps can be regarded as the maps on cohomology induced by the inclusion $\partial\bar{M} \hookrightarrow \bar{M}$. Thus, the proposition below is a dual version of Corollaire 3.23 in [46].

Proposition 3.4.2. *The map* $r^2\colon H^2(M;\mathscr{K}) \to H^2(M;\mathscr{K}_{\partial\bar{M}})$ *is an isomorphism.*

Proof. The relevant piece of the long exact sequence in cohomology associated to the short exact sequence of sheaves $0 \to \mathscr{K}_{\bar{M},\partial\bar{M}} \to \mathscr{K} \to \mathscr{K}_{\partial\bar{M}} \to 0$ is

$$\cdots \to H^2(\bar{M};\mathscr{K}) \xrightarrow{r^2} H^2(\bar{M};\mathscr{K}_{\partial\bar{M}}) \to H^3(\bar{M};\mathscr{K}_{\bar{M},\partial\bar{M}}) \to \cdots$$

Poincaré duality implies $H^3(\bar{M};\mathscr{K}_{\bar{M},\partial\bar{M}}) \cong H^0(M;\mathscr{K})$, with the latter vanishing by Lemma 3.4.1. Hence, r^2 is an epimorphism. But $\dim H^2(\bar{M};\mathscr{K}_{\partial\bar{M}}) = k$, so r^2 must be an isomorphism. Q.E.D.

3.5 Definition of the non-abelian torsion

We continue using the notations of the previous section. Suppose that $\underline{h}$ is a basis of the cohomology group $H^\bullet(M;\mathscr{K}) = H^1(M;\mathscr{K}) \oplus H^2(M;\mathscr{K})$. We may assume $\underline{h} = \underline{h^1} \sqcup \underline{h^2}$ where h^1 is a basis of $H^1(M;\mathscr{K})$ and h^2 is a basis of $H^2(M;\mathscr{K})$. Recall that, by Lemma 3.4.1, these spaces have equal dimension, equal to the number k of cusps of M. Furthermore, the above remains true when the complete structure on M is deformed into incomplete structures [46].

Definition 3.5.1 (Balanced basis)**.** Suppose that $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multicurve satisfying Assumption 1.2.1. We say that a basis $\underline{h} = \underline{h^1} \sqcup \underline{h^2}$ of $H^\bullet(M;\mathscr{K})$ is *balanced with respect to the multicurve* γ if

$$r^2(\underline{h^2}) = \{\, x \smile [\gamma]^* \mid x \in r^1(\underline{h^1}) \},\tag{3.5.1}$$

where $[\gamma]^* \in H^1(\partial\bar{M};\mathbb{C})$ is the Poincaré dual of the homology class of γ in $H_1(\partial\bar{M};\mathbb{C})$.

The above definition makes use of the cup product

$$\smile\colon H^1(\partial\bar{M};\mathscr{K}_{\partial\bar{M}}) \times H^1(\partial\bar{M};\mathbb{C}) \to H^2(\partial\bar{M};\mathscr{K}_{\partial\bar{M}}),$$

which exists since $\mathscr{K}$ is a sheaf of complex vector spaces, so $\mathscr{K} \otimes \mathbb{C} = \mathscr{K}$.

It is not clear from our definition that a balanced basis always exists. In general, the existence follows from [46, Proposition 4.5] and can also be concluded from Corollary 4.4.4 below.

Definition 3.5.2 (The non-abelian Reidemeister torsion of the adjoint holonomy representation)**.** Given a pair (M, γ) where M is a connected, complete hyperbolic 3-manifold of finite volume with k cusps equipped with an arbitrary finite cell decomposition and γ is a multicurve satisfying conditions of Definition 3.5.1, we define

$$\mathbb{T}_{\mathrm{Ad}}(M, \gamma) \stackrel{\mathrm{def}}{=\!=} \mathbb{T}\,(C^\bullet(M; \mathrm{Ad} \circ \mathrm{hol}), \underline{c}, \underline{h}) \in \mathbb{C}_\times / \{\pm 1\},$$

where $C^\bullet(M; \mathrm{Ad} \circ \mathrm{hol})$ is the twisted cellular cochain complex of Definition 3.3.1, $\underline{c}$ is any geometric basis of $C^\bullet(M; \mathrm{Ad} \circ \mathrm{hol})$ and $\underline{h}$ is any cohomology basis balanced with respect to γ.

We know from Proposition 3.3.7 that the value of the torsion does not depend on the choice of a geometric basis. However, in order to show that $\mathbb{T}_{\mathrm{Ad}}(M, \gamma)$ is well-defined, we still need to prove that any two cohomology bases balanced with respect to the same multicurve γ give the same torsion.

Suppose $\underline{h}$ and $\underline{\hat{h}}$ are both balanced with respect to γ. Define a map

$$b_\gamma \colon H^1(M; \mathscr{K}) \to H^2(M; \mathscr{K}_{\partial \bar{M}}), \quad b_\gamma(x) = r^1(x) \smile [\gamma]^*.$$

By Proposition 3.4.2, $r^2 \colon H^2(M; \mathscr{K}) \to H^2(M; \mathscr{K}_{\partial \bar{M}})$ is an isomorphism, so we have

$$\det\left[\underline{\hat{h}^2} / \underline{h^2}\right] = \det\left[r^2(\underline{\hat{h}^2}) / r^2(\underline{h^2})\right] = \pm \det\left[b_\gamma(\underline{\hat{h}^1}) / b_\gamma(\underline{h^1})\right],$$

where the last equality follows from the assumption that $\underline{h}$ and $\underline{\hat{h}}$ are balanced; the sign indeterminacy stems from the fact that the sets on the two sides of the equality (3.5.1) may be ordered differently. It follows that b_γ is also an isomorphism and the right hand-side equals $\pm \det\left[\underline{\hat{h}^1} / \underline{h^1}\right]$. Theorem 3.1.4 now implies that the two torsions are equal.

Finally, we see that the only reason to use the complete hyperbolic structure on M is to ensure that $\mathbb{T}_{\mathrm{Ad}}(M, \gamma)$ is a uniquely defined topological invariant. If we deform the complete structure to an incomplete one, Porti's construction still carries through, as explained in detail in [46, Ch. 4]. Hence, we can more generally view $\mathbb{T}_{\mathrm{Ad}}(M, \gamma)$ as a complex-valued function, defined up to sign, on the deformation space of hyperbolic structures. Algebraically speaking, the point in the $PSL_2\mathbb{C}$-character variety of M corresponding to the discrete faithful representation has an open neighbourhood on which the torsion is an analytic function. We postpone the more detailed discussion of the character variety until Section 4.2.

3.6 Geometric bases of Čech cochain complexes

We proceed to restate the definition of a *geometric basis* (Definition 3.3.2) in terms of the flat vector bundle $E = E_\varrho$ determined by an n-dimensional unimodular representation ϱ. Roughly speaking, geometric bases correspond to constant, non-zero sections of the (trivial) determinant bundle $\det E$. This coordinate-free interpretation of Porti's normalization is then applied to Čech cochains with coefficients in the sheaf of locally constant (parallel) sections of E, which is the content of Theorem 3.6.7. In this way, we can generalize the concept of a geometric basis to Čech cochain complexes with coefficients in a flat vector bundle of trivial determinant.

Subsequently, we specialize to the case $n = 3$ which is of particular relevance to the non-abelian Reidemeister torsion $\mathbb{T}_{\mathrm{Ad}}(M,\gamma)$ of a hyperbolic 3-manifold M. In this case, the 3-dimensional complex unimodular representation ϱ is given as the adjoint of the hyperbolic holonomy representation of M. As discussed in Section 2.3.3, this representation can be understood in terms of analytic continuation of Killing fields, so our approach produces geometric bases of Čech cochain complexes with coefficients in $\mathscr{K}$ by purely geometric means.

3.6.1 Coordinate-free description of geometric bases

Given a based topological space $(X,*)$ with the homotopy type of a finite path-connected cell complex, any n-dimensional representation $\varrho\colon \pi_1(X,*) \to GL(n,\mathbb{C})$ determines a vector bundle $E = E_\varrho \xrightarrow{\pi} X$ whose fibres are n-dimensional vector spaces over $\mathbb{C}$. Denoting by $\widetilde{X}$ the universal covering space of X, we can explicitly define E as

$$E = \widetilde{X} \times \mathbb{C}^n \Big/ (\gamma x,\, \varrho(\gamma)v) \sim (x,v), \quad \gamma \in \pi_1(X,*), \tag{3.6.1}$$

where $x \mapsto \gamma x$ is the action of the element γ on $\widetilde{X}$ by a deck transformation. As mentioned in the opening of Chapter 2, there are two common ways of topologizing the bundle E.

Firstly, we can endow the fibres of E with the discrete topology. This makes $E \xrightarrow{\pi} X$ an *étale bundle*, i.e., π becomes a local homeomorphism. The sheaf $\mathscr{F}^{\text{ét}}$ of continuous sections of E with discrete topology is then automatically a locally constant sheaf.

Alternatively, when X is a smooth manifold, we can endow the fibres of E with the Euclidean topology and the usual smooth structure on $\mathbb{C}^n$, turning the total space E into a smooth manifold. Furthermore, the smooth bundle $E \xrightarrow{\pi} X$ admits a canonical flat connection ∇. For an open set $U \subseteq X$, a smooth section $\sigma\colon U \to E$ of this bundle is called *parallel* if $\nabla_Y \sigma = 0$ for every vector field Y on U. The sheaf $\mathscr{F}^{\nabla=0}$ of parallel sections of E is again a locally constant sheaf, isomorphic to the sheaf $\mathscr{F}^{\text{ét}}$.

In this section, we shall use the term "flat bundle" to refer to either possible topology on E. In the subsequent section, we shall focus on the study of the sheaf $\mathscr{F}^{et} \cong \mathscr{F}^{\nabla=0}$. The isomorphism class of the bundle E, both as an étale bundle as well as a smooth bundle with a flat connection, depends only on the conjugacy class of ϱ.

For the remainder of this section, let $E = E_\varrho$ be the flat bundle determined by ϱ. Consider moreover the flat bundle $\det E$ corresponding to the composite representation

$$\det\varrho\colon \pi_1(X,*) \xrightarrow{\varrho} GL(n,\mathbb{C}) \xrightarrow{\det} \mathbb{C}_\times = GL(1,\mathbb{C}).$$

For a *unimodular* representation ϱ, the representation $\det\varrho$ is trivial and therefore the bundle $\det E$ is the trivial bundle $X \times \mathbb{C}$. More concretely, the total space of $\det E$ is given by

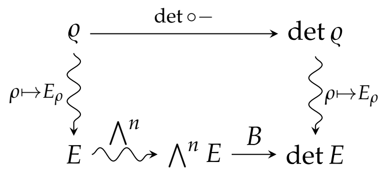
$$\det E = \widetilde{X} \times \mathbb{C} \Big/ (\gamma x,\, \det\varrho(\gamma)z) \sim (x,z), \quad \gamma \in \pi_1(X,*).$$

If $\det\varrho \equiv 1$, the equivalence relation becomes $(\gamma x, z) \sim (x,z)$, giving

$$\det E = X \times \mathbb{C}. \tag{3.6.2}$$

On the other hand, we can consider the top exterior power $\bigwedge^n E$ of the bundle E. As explained in [3, §1.2], $\bigwedge^n E$ is a line bundle whose fibre over a point $x \in X$ is the top exterior power $\bigwedge^n E_x$ of the fibre E_x of the bundle E.

Even without the assumption of unimodularity, there is always an isomorphism of flat bundles $\bigwedge^n E \cong \det E$, albeit not canonical. We are going to describe an elementary way of fixing one such isomorphism in the case when ϱ is unimodular. This will lead us to a description of a bundle isomorphism B completing the following diagram.



In the above diagram, the bottom row consists of vector bundles on X. The nth exterior power $\bigwedge^n$ is an endofunctor on the category of vector bundles. The vertical wavy arrows refer to the construction of flat bundles from linear representations of $\pi_1(X)$ by the way of (3.6.1).

Suppose $\underline{e} = \{e_1, e_2, \dots, e_n\} \subseteq E_*$ is an ordered basis of the fibre of E at the basepoint. Taking an exterior product, we obtain the element

$$\mathrm{vol}(\underline{e}) := e_1 \wedge e_2 \wedge \cdots \wedge e_n \in \bigwedge^n E_*.$$

Since $\underline{e}$ is a basis, $\mathrm{vol}(\underline{e}) \neq 0$.

Proposition 3.6.1. *If X is a finite path-connected CW-complex, then there exists an isomorphism of flat bundles*

$$B\colon \bigwedge^n E \xrightarrow{\cong} \det E.$$

If furthermore ϱ is unimodular, there is a unique such isomorphism satisfying $B(\mathrm{vol}(\underline{e})) = 1$.

Proof. Consider $\widetilde{B} := (\mathrm{Id} \times \det)\colon \widetilde{X} \times \bigwedge^n \mathbb{C}^n \to \widetilde{X} \times \mathbb{C}$. Note that the action of an element $\gamma \in \pi_1(X)$ on $\bigwedge^n \mathbb{C}^n$ is by multiplication with $\det \varrho(\gamma)$. This shows that $\widetilde{B}$ is equivariant with respect to the action of the fundamental group. Therefore, $\widetilde{B}$ descends to the quotient, inducing a well-defined bundle isomorphism $B\colon \bigwedge^n E \to \det E$.

Assume now that ϱ is unimodular and observe that (3.6.2) allows us to identify the number $1 \in \mathbb{C}$ with the global section $x \mapsto (x, 1)$ of $\det E$. Let $\widetilde{*}$ be a lift of the basepoint $*$ to $\widetilde{X}$. There is a unique vector $v \in \bigwedge^n \mathbb{C}^n$ such that $[(\widetilde{*}, v)] = \mathrm{vol}(\underline{e}) \in \bigwedge^n E_*$, where $[\ ]$ denotes the orbit of the action of the fundamental group. Multiplying $\widetilde{B}$ by a constant if necessary, we can arrange so that

$$\widetilde{B}(\widetilde{*}, v) = (\widetilde{*}, 1). \tag{3.6.3}$$

This ensures that B maps $\mathrm{vol}(\underline{e})$ to 1. To prove uniqueness, it suffices to notice that the assumption of connectedness of X implies that two constant maps on X are equal if they are equal at a single point. Hence, if $B(\mathrm{vol}(\underline{e})) = B'(\mathrm{vol}(\underline{e})) = 1$, we have the equality of the global pull-back sections $B^*1 = B'^*1$. Since $\bigwedge^n E$ has rank one, this implies $B = B'$. Q.E.D.

For the remainder of this section, we assume ϱ to be unimodular. The above proposition shows that a global trivialization $B = B(\underline{e})$ is uniquely determined by an ordered basis $\underline{e} \subseteq E_*$ at the basepoint. In case the basis $\underline{e}$ is not ordered, the element $\mathrm{vol}(\underline{e})$ is only defined up to sign and consequently B is unique only up to sign.

Definition 3.6.2. Suppose $E = E_\varrho \to X$ is the bundle associated to an n-dimensional unimodular representation ϱ and that the isomorphism $B\colon \bigwedge^n E \xrightarrow{\cong} X \times \mathbb{C}$ is determined by $\underline{e} \subseteq E_*$ as in Proposition 3.6.1. For any $x \in X$, a (not necessarily ordered) basis $\underline{f} = \{f_1, \dots, f_n\} \subseteq E_x$ is said to be *compatible with* $\underline{e}$ if $B(\bigwedge_{i=1}^n f_i) = \pm 1$.

In other words, a compatible basis agrees up to sign with the unique global, constant section of the trivial bundle $\bigwedge^n E$ whose value at $*$ is $\mathrm{vol}(\underline{e})$.

3.6.2 Normalization of bases of Čech cochain complexes

In this section, we explain how Definition 3.6.2 applies to the choice of bases of Čech cochain spaces. A detailed treatment of Čech cohomology with coefficients in an arbitrary sheaf of abelian groups can be found in the book by R. Godement [20]; for a good crash course we refer to J. Hubbard [28, Appendix A7]. Herein, we limit ourselves to the study of cohomology with coefficients in a local system (i.e., a locally constant sheaf) of complex vector spaces. We assume, for the entirety of this section, that X is a topological space with the homotopy type of a finite, connected CW-complex and that the bundle $E = E_\varrho$ on X is given by a complex unimodular representation of $\pi_1(X)$. In particular, all results of this section apply when X is a connected 3-manifold.

For an open set $U \subseteq X$, denote by $\mathscr{F}(U)$ the set of all locally constant (parallel) sections of E on U. The assignment

$$U \mapsto \mathscr{F}(U)$$

defines a sheaf $\mathscr{F}$ called the *sheaf of locally constant sections of* E. Whenever $V \subseteq U$, we have the restriction operator

$$r_V^U\colon \mathscr{F}(U) \to \mathscr{F}(V), \qquad r_V^U(\sigma) = \sigma|_V.$$

The restriction operators are linear and thus $\mathscr{F}$ is in fact a sheaf of complex vector spaces.

For a good open cover $\mathfrak{U}$ of X, we would like to use the Čech cochain complex $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ to calculate the Reidemeister torsion of X with coefficients given by ϱ. In order to do this, we need to be able to extend the concept of a geometric basis (Definition 3.3.2) to work with Čech cochains.

Definition 3.6.3. Let $\mathscr{F}$ be the sheaf of sections of E and let $\underline{e} \subseteq E_*$ be any basis. For a simply connected open set $U \subseteq M$, a basis $\underline{f}$ of $\mathscr{F}(U)$ is said to be *compatible with* $\underline{e}$ if either $U = \varnothing$ or there exists a point $x \in U$ such that $(\underline{f})_x \subset E_x$ is compatible with $\underline{e}$ in the sense of Definition 3.6.2.

Remark 3.6.4. We remark that if the condition in the above definition is satisfied for any $x \in U$, then it is automatically satisfied for every $x \in U$. Indeed, consider two points $x, y \in U$. Since U is simply connected, any two paths in U from x to y are homotopic rel $\{x, y\}$ and thus the monodromy $t_U\colon E_x \to E_y$ within U is uniquely defined and fits into the commutative diagram

$$\begin{array}{ccc} & \mathscr{F}(U) & \\ {}^{\mathrm{ev}_x}\swarrow & & \searrow^{\mathrm{ev}_y} \\ E_x & \xrightarrow{\quad t_U \quad} & E_y, \end{array} \tag{3.6.4}$$

in which the diagonal arrows are given by evaluation of sections. Given a basis $\underline{f} \subseteq \mathscr{F}(U)$, we get

$$\bigwedge \mathrm{ev}_y(\underline{f}) = \pm \det t_U \left(\bigwedge \mathrm{ev}_x(\underline{f})\right).$$

Applying the map B of Proposition 3.6.1, we obtain

$$B\left(\bigwedge_{i=1}^{n} f_i(y)\right) = \pm B\left(\bigwedge_{i=1}^{n} f_i(x)\right),$$

where we have used (3.6.2) to compare vectors in fibres over x and y.

We now proceed to extend the notion of a geometric basis to Čech cochain spaces. Recall that if $\mathfrak{U}$ is an open cover of M, every d-simplex Δ of the nerve $\mathcal{N}(\mathfrak{U})$ is associated to a non-empty intersection

$$U_\Delta = \bigcap_{i=0}^{d} U_i \tag{3.6.5}$$

of certain open sets $U_i \in \mathfrak{U}$. The set U_Δ is called the *carrier* of the simplex Δ in [14, p. 234]. The dth Čech cochain space of $\mathfrak{U}$ is defined as

$$\check{C}^d(\mathfrak{U}, \mathscr{F}) = \prod_{\Delta} \mathscr{F}(U_\Delta), \tag{3.6.6}$$

where Δ runs over the set of all oriented d-simplices of $\mathcal{N}(\mathfrak{U})$. When $\mathfrak{U}$ is a good open cover, all of the carrier sets U_Δ are contractible. Since (3.6.6) is essentially a special case of (3.3.2), we may adopt the following definition.

Definition 3.6.5 (Čech geometric basis)**.** Suppose $\mathfrak{U}$ is a finite good open cover of X and $\mathscr{F}$ is the sheaf of locally constant sections of the bundle E with an arbitrarily fixed basis $\underline{e} \subseteq E_*$. For every simplex Δ of $\mathcal{N}(U)$, equip the space $\mathscr{F}(U_\Delta)$ with a basis $\underline{f_\Delta}$ compatible with $\underline{e}$ in the sense of Definition 3.6.3. Any basis of $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ obtained as the union of such bases over all simplices of $\mathcal{N}(\mathfrak{U})$ is called a *Čech geometric basis*.

Remark 3.6.6. We stress that the notion of a Čech geometric basis defined above depends on the choice of a basis $\underline{e}$ for E_*.

At present, we are going to show that a *Čech geometric basis* of Definition 3.6.5 is indeed a Čech cochain equivalent of a geometric basis in the sense of Definition 3.3.2. We note that the nerve $\mathcal{N}(\mathfrak{U})$ of a good open cover $\mathfrak{U}$ of X has the same homotopy type as X. In particular, Čech cohomology of $\mathfrak{U}$ computes the simplicial (and hence cellular) cohomology of X, as explained in [28, p. 389].

According to [20, Section 5.1], the Čech cochain complex $\check{C}(\mathfrak{U}, \mathscr{F})$ is isomorphic to the simplicial (hence cellular) cochain complex $C^\bullet(\mathcal{N}(\mathfrak{U}); \varrho)$. We are going to briefly explain how to write down such an isomorphism. Fix a degree $d \geq 0$. Given an oriented d-simplex Δ of $\mathcal{N}(\mathfrak{U})$, let $\widetilde{\Delta}$ denote an arbitrarily chosen, compatibly oriented lift of Δ to $\widetilde{\mathcal{N}(\mathfrak{U})}$ understood as a cell complex. The pre-image $\pi^{-1}(U_\Delta)$ of U_Δ under the universal covering projection consists, on the level of $\mathcal{N}(\mathfrak{U})$, of all possible lifts of the simplex Δ, which correspond bijectively to the connected components of $\pi^{-1}(U_\Delta)$. Hence, the chosen lift $\widetilde{\Delta}$ corresponds to a particular connected component $\widetilde{U_\Delta} \subseteq \pi^{-1}(U_\Delta)$ which is taken by the covering map π homeomorphically onto U_Δ.

In particular, for every $x \in U_\Delta$ there is a unique lift $\widetilde{x} \in \widetilde{U_\Delta}$. We are going to think of $\widetilde{M}$ as of the space of all paths in M ending at the basepoint, taken up to homotopy fixing the endpoints.

In this way, we can associate to the lift $\widetilde{x}$ a path $\gamma_{\widetilde{x}}\colon [0,1] \to M$ satisfying $\gamma_{\widetilde{x}}(0) = x$, $\gamma_{\widetilde{x}}(1) = *$. Given a fixed isomorphism $\varphi\colon E_* \xrightarrow{\cong} \mathbb{C}^n$, we define a map

$$\beta_d\colon \check{C}^d(\mathfrak{U},\mathscr{F}) \to C^d(\mathcal{N}(\mathfrak{U}),\varrho),$$

by setting, for any section $\sigma \in \mathscr{F}(U_\Delta)$,

$$\beta_d(\sigma)(\widetilde{\Delta}) = \varphi\left(t_{\gamma_{\widetilde{x}}}(\sigma(x))\right) \tag{3.6.7}$$
$$\text{and } \beta_d(\sigma)(\Gamma) = 0 \text{ when } \Gamma \text{ is not a lift of } \Delta,$$

where $t_{\gamma_{\widetilde{x}}}$ is the monodromy of $\mathscr{F}$ along the path $\gamma_{\widetilde{x}}$. This monodromy only depends on the lift $\widetilde{\Delta}$ used on the left-hand side of (3.6.7). Furthermore, (3.6.4) shows that β_d does not depend on the choice of the point $x \in U_\Delta$.

Recall that (3.6.6) and (3.3.2) give decompositions of the domain and the codomain of β_d into products of subspaces associated to the oriented simplices. By definition, the map β_d preserves these decompositions, i.e.,

$$\beta_d\left(\mathscr{F}(U_\Delta)\right) = V(\Delta) \tag{3.6.8}$$

for any d-simplex Δ. It is easy to see that $\beta = \bigoplus_d \beta_d$ is the sought for isomorphism between $\check{C}(\mathfrak{U},\mathscr{F})$ and $C^\bullet(\mathcal{N}(\mathfrak{U});\varrho)$.

Theorem 3.6.7. *Suppose that $\mathscr{F}$ is the sheaf of locally constant sections of E and that $\underline{e} \subseteq E_*$ is a fixed basis. Consider the Čech cochain complex $\check{C}^\bullet(\mathfrak{U},\mathscr{F})$ and choose a basis $\underline{h}$ for the cohomology $H^\bullet(M,\mathscr{F}) = \check{H}^\bullet(\mathfrak{U},\mathscr{F})$. Let $\underline{\check{c}} = \underline{\check{c}}(\underline{e})$ be a Čech geometric basis of $\check{C}^\bullet(\mathfrak{U},\mathscr{F})$. Let furthermore $\underline{c}$ be a geometric basis for the cellular cochain complex $C^\bullet(\mathcal{N}(\mathfrak{U}),\varrho)$ defined with the help of a basis $\underline{b} \subseteq \mathbb{C}^n$ satisfying* $\mathrm{vol}(\underline{b}) = \pm 1$. *We have*

$$\mathbb{T}(\check{C}^\bullet(\mathfrak{U},\mathscr{F}),\underline{\check{c}},\underline{h}) = \mathbb{T}(C^\bullet(\mathcal{N}(\mathfrak{U}),\varrho),\underline{c},\underline{h}).$$

Proof. Any ordering of the basis $\underline{e} \subseteq E_*$ determines an isomorphism $\varphi\colon E_* \xrightarrow{\cong} \mathbb{C}^n$ given by the coordinate map; changing the order affects φ but $\det\varphi$ remains constant up sign. The idea of proof is now to use this φ in (3.6.7) and to understand, for each d, the change-of-basis matrix relating the basis $\beta_d(\underline{\check{c}^d})$ to the basis $\underline{c^d}$. According to Definition 3.6.5, the basis $\underline{\check{c}^d}$ is a union of bases of the spaces $\mathscr{F}(U_\Delta)$ associated by $\mathscr{F}$ to the carriers of all d-simplices Δ. Since the map β_d satisfies the diagonality condition (3.6.8), it suffices to consider the effect of β_d on the component $\mathscr{F}(U_\Delta)$ for each d-simplex Δ.

Given a d-simplex Δ, let $\underline{f_\Delta} = \underline{\check{c}^d} \cap \mathscr{F}(U_\Delta) = \{f^i_\Delta\}_{i=1}^n$ be the part of $\underline{\check{c}^d}$ corresponding to the factor $\mathscr{F}(U_\Delta)$. Similarly, let $\{c^{(i)}_\Delta\}_{i=1}^n$ be the part of the geometric basis $\underline{c^d}$ spanning the subspace $V(\Delta)$ of $C^d(\mathcal{N}(\mathfrak{U}),\varrho)$. With the above notations, we have

$$\det\left[\beta_d(\underline{\check{c}^d})/\underline{c^d}\right] = \pm \prod_{\Delta\in S^{(d)}} \det\left[\beta_d(\underline{f_\Delta})/\{c^{(i)}_\Delta\}_{i=1}^n\right], \tag{3.6.9}$$

where $S^{(d)}$ is the finite set of all d-simplices of $\mathcal{N}(\mathfrak{U})$.

Note that due to the sign indeterminacy already inherent to the determinants of change-of-basis matrices, we may ignore the orientations of simplices. According to (3.6.7),

$$\beta_d(f^i_\Delta)(\widetilde{\Delta}) = \varphi\left(t_{\gamma_{\tilde{x}}}\left(f^i_\Delta(x)\right)\right),$$

where $x \in U_\Delta$ and $\widetilde{\Delta}$ is any lift of Δ to the universal covering space. Taking the exterior product of these elements over all $i \in \{1, \ldots, n\}$, we get

$$\det \varphi^{-1} \bigwedge_{i=1}^{n} \beta_d(f^i_\Delta)(\widetilde{\Delta}) = \det t_{\gamma_{\tilde{x}}} \left(\bigwedge_{i=1}^{n} f^i_\Delta(x)\right). \tag{3.6.10}$$

According to Definition 3.6.3, our assumption on $\underline{\check{c}}$ means that the basis $\mathrm{ev}_x(\underline{f_\Delta}) = \{f^i_\Delta(x)\}_{i=1}^n$ is compatible with $\underline{e}$; thus

$$B\left(\det \varphi^{-1} \bigwedge_{i=1}^{n} \beta_k(f^i_\Delta)(\widetilde{\Delta})\right) = B\left(\det t_{\gamma_{\tilde{x}}} \bigwedge_{i=1}^{n} f^i_\Delta(x)\right) = \pm 1,$$

where B is constructed in Proposition 3.6.1. Note that this statement is true for every lift $\widetilde{\Delta}$, since any two lifts are related by a deck transformation and $\pi_1(X)$ acts trivially on $\bigwedge^n \mathbb{C}^n$ as a consequence of unimodularity of ϱ. Since $\det \varphi^{-1}\colon \bigwedge^n \mathbb{C}^n \to \bigwedge^n E_*$ satisfies $\det \varphi^{-1}(1) = \pm \operatorname{vol}(\underline{e})$ and $B(\operatorname{vol}(\underline{e})) = \pm 1$, we must have

$$\bigwedge_{i=1}^{n} \beta_d(f^i_\Delta)(\widetilde{\Delta}) = \pm 1. \tag{3.6.11}$$

Similarly, the assumption on $\underline{b}$ yields

$$\bigwedge_{i=1}^{n} c^{(i)}_\Delta(\widetilde{\Delta}) = \pm \bigwedge_{i=1}^{n} b_i \overset{\text{def}}{=\!=} \pm \operatorname{vol}(\underline{b}) = \pm 1 \tag{3.6.12}$$

as long as $\widetilde{\Delta}$ is the lift of Δ used in the definition of $c^{(i)}_\Delta$. Using a different lift $\gamma(\widetilde{\Delta})$, where $\gamma \in \pi_1(X)$, introduces a factor of $\det \varrho(\gamma) = 1$, thus (3.6.12) holds regardless of the choice of lifts.

Equalities (3.6.11) and (3.6.12) show that $\beta_d(\underline{f_\Delta})$ and $\{c^{(i)}_\Delta\}_{i=1}^n$ determine the same element of $\bigwedge^n V(\Delta)$; therefore, each term in the product (3.6.9) is equal to 1 or -1. Using Theorem 3.1.4, we conclude that the torsions of both complexes are equal. Q.E.D.

Corollary 3.6.8. *If $\chi(X) = 0$, Theorem 3.6.7 holds without any assumption on the basis $\underline{b} \subseteq \mathbb{C}^n$.*

Proof. Proposition 3.3.7 states that the torsion $\mathbb{T}(C^\bullet(\mathcal{N}(\mathfrak{U}), \varrho), \underline{c}, \underline{h})$ does not depend on the choice of the geometric basis $\underline{c}$. Therefore any basis $\underline{b}$ of $\mathbb{C}^n$ can be used to define $\underline{c}$. Q.E.D.

3.6.3 Normalization of the adjoint non-abelian torsion via geometric atlases

In this section, we explain how the ideas developed so far in this chapter apply to the study of the non-abelian Reidemeister torsion of a complete finite-volume hyperbolic 3-manifold M. In particular, we explain why a geometric basis is called *geometric*. In this section we set

$$\varrho := \mathrm{Ad} \circ \mathrm{hol}\colon \pi_1(M) \to \mathrm{Aut}_{\mathbb{C}}(\mathfrak{sl}_2\mathbb{C}), \tag{3.6.13}$$

where hol is a holonomy representation of the hyperbolic structure on M.

If we identify the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ with $\mathbb{C}^3$ by the means of any linear isomorphism $\varphi\colon \mathfrak{sl}_2\mathbb{C} \xrightarrow{\cong} \mathbb{C}^3$, then ϱ gives a homomorphism $\pi_1(M) \to SL(3,\mathbb{C})$, because the adjoint representation of $PSL_2\mathbb{C}$ is unimodular. In this way, the discussion of the preceding sections applies to ϱ, with the sheaf $\mathscr{K}$ playing the role of $\mathscr{F}$.

Let $\mathfrak{A} = \{(U_i,\ \Phi_i\colon U_i \to \mathbb{H}^3)\}_{i\in I}$ be an orientation-preserving geometric atlas on M. In other words, we assume that the open sets U_i cover M and for every i, $\Phi_i\colon U_i \to \mathbb{H}^3$ is an orientation-preserving hyperbolic isometry onto its image. Assume furthermore that the index set I is finite. Of special interest to us will be the situation when the open cover $\mathfrak{U}(\mathfrak{A}) := \{U_i\}_{i\in I}$ is a good open cover, because we would like to apply Theorem 3.6.7 to $\mathfrak{U}(\mathfrak{A})$ and the sheaf $\mathscr{K}$. To this end, we need to construct a Čech geometric basis of the corresponding Čech cochain complex. The general method is to push forward local Killing vector fields on $\mathbb{H}^3$ to M and use the induced identifications of the spaces of local sections of $\mathscr{K}$ on M with the Lie algebra $\mathfrak{sl}_2\mathbb{C}$.

Assume that $X \in \mathscr{K}(\mathbb{H}^3)$ is a Killing vector field on $\mathbb{H}^3$ and $\Phi\colon U \to \mathbb{H}^3$ is one of the charts of $\mathfrak{A}$. Then the push-forward field $(\Phi^{-1})_*X$ is a Killing field on U. In this way, we have a map

$$\mathscr{K}(\mathbb{H}^3) \xrightarrow{(\Phi^{-1})_*} \mathscr{K}(U), \tag{3.6.14}$$

where on both sides $\mathscr{K}$ denotes the sheaves of Killing vector fields on the respective hyperbolic manifolds. Moreover, the above map is an isomorphism. We refer to Section 1.1 and to [43, 34] for more details. Thus, the atlas $\mathfrak{A}$ induces a collection of isomorphisms given by compositions of the form

$$\psi_U\colon \mathfrak{sl}_2\mathbb{C} \xrightarrow{\psi} \mathscr{K}(\mathbb{H}^3) \xrightarrow{(\Phi_i^{-1})_*} \mathscr{K}(U), \quad U \in \mathfrak{U}(\mathfrak{A}), \tag{3.6.15}$$

where ψ is the map of (1.1.3). We are going to use the above isomorphisms ψ_U, $U \in \mathfrak{U}(\mathfrak{A})$, to construct a Čech geometric basis in the sense of Definition 3.6.5. We shall work with ordered Čech cochains and write $\mathfrak{U}$ for $\mathfrak{U}(\mathfrak{A})$. Let $\prec$ be a total order on the set $\mathfrak{U}$. Thus, every oriented d-simplex Δ of $\mathcal{N}(\mathfrak{U})$ corresponds to a finite chain $V_0 \prec V_1 \prec \cdots \prec V_d$ satisfying

$$V_\Delta := \bigcap_{i=0}^{d} V_i \neq \varnothing. \tag{3.6.16}$$

Let $\underline{b} = \{b_1, b_2, b_3\}$ be a fixed basis of $\mathfrak{sl}_2\mathbb{C}$. For any $V \in \mathfrak{U}$, we equip the space $\mathscr{K}(V)$ with the basis $\psi_V(\underline{b})$. For each open set V_Δ of (3.6.16), consider $\underline{f^0_\Delta} := r^{V_0}_{V_\Delta} \circ \psi_{V_0}(\underline{b}) \subset \mathscr{K}(V_0 \cap V_\Delta)$. It is clear that $\underline{f^0_\Delta}$ is a basis for $\mathscr{K}(V_0 \cap V_\Delta)$. Since V_Δ is simply-connected, the Killing fields in $\underline{f^0_\Delta}$ admit unique analytic continuations onto V_Δ. These analytic continuations remain linearly independent and therefore form a basis $\underline{f_\Delta}$ of the space $\mathscr{K}(V_\Delta)$. Repeating this construction for all simplices Δ of $\mathcal{N}(\mathfrak{U})$, we obtain a basis

$$\mathfrak{A}(\underline{b}) := \bigcup_{\Delta} \underline{f_\Delta} \tag{3.6.17}$$

of the Čech cochain complex $\check{C}^\bullet(\mathfrak{U}, \mathscr{K})$.

Remark 3.6.9. Note that the choice of the ordering $\prec$ determines which of the sets in $\mathfrak{U}$ is the minimal element in the chain corresponding to an oriented simplex Δ, thus it determines which of the sets will play the role of the V_0's in the above construction of $\mathfrak{A}(\underline{b})$.

Theorem 3.6.10. *Let M be a connected, oriented, geometrically finite hyperbolic 3-manifold admitting a finite orientation-preserving geometric atlas $\mathfrak{A}$ whose coordinate charts form a good open cover $\mathfrak{U} = \mathfrak{U}(\mathfrak{A})$ of M. If $\underline{b} \subset \mathfrak{sl}_2\mathbb{C}$ is an arbitrarily fixed basis, then $\mathfrak{A}(\underline{b})$ is a Čech geometric basis of $\check{C}^\bullet(\mathfrak{U}, \mathscr{K})$.*

Proof. By Remark 3.6.4, it suffices to show that for every simplex Δ of $\mathcal{N}(\mathfrak{U})$, the basis $\underline{f_\Delta}$ occurring in (3.6.17) satisfies $B(\mathrm{vol}(\underline{f_\Delta}(x))) = \pm 1$, where x is any point in the carrier set V_Δ of (3.6.16) and B is given by Proposition 3.6.1.

The bundle $E = E_\varrho$ given by the adjoint representation ϱ of (3.6.13) can be constructed as

$$E = \widetilde{M} \times \mathfrak{sl}_2\mathbb{C} \Big/ (\gamma x,\, \varrho(\gamma)v) \sim (x, v), \quad \gamma \in \pi_1(M, *).$$

At first, we shall set $* = x$. Write $\underline{b} = \{b_1, b_2, b_3\}$ and consider the elements $(\widetilde{*}, b_i) \in \widetilde{M} \times \mathfrak{sl}_2\mathbb{C}$ where $\widetilde{*}$ is any fixed lift of $*$ and $i \in \{1, 2, 3\}$. Set $\underline{e} = \{[(\widetilde{*}, b_i)]_\sim\}_{i=1,2,3}$. Recall that the basis $\underline{e}$ of E_* determines the bundle isomorphism B defined as the quotient of the unique flat bundle map $\widetilde{B}$ satisfying the normalization condition (3.6.3). In our setting, $\widetilde{B}\colon \widetilde{M} \times \bigwedge^3 \mathfrak{sl}_2\mathbb{C} \to \widetilde{M} \times \mathbb{C}$ and the condition (3.6.3) becomes

$$\widetilde{B}(\widetilde{*}, b_1 \wedge b_2 \wedge b_3) = (\widetilde{*}, 1). \tag{3.6.18}$$

Given $\Delta = (V_0 \prec V_1 \prec \cdots \prec V_k)$, write $\underline{f_\Delta} = \{f_1, f_2, f_3\}$. Since $* \in V_\Delta \cap V_0$, we have

$$f_i(*) = f_i^0(*) = \left((r^{V_0}_{V_\Delta} \circ \psi_{V_0})(b_i)\right)(*) = \psi_{V_0}(b_i)(*), \tag{3.6.19}$$

and consequently $B(f_1(*) \wedge f_2(*) \wedge f_3(*)) = \pm D$, where D is the determinant of ψ_{V_0} with respect to the bases $\underline{b}$ and $\underline{f_\Delta}$. But (3.6.19) says that ψ_{V_0} relates these two bases, so $D = 1$.

It is easy to see that the above remains true without the assumption $* = x$. In fact, since $\widetilde{B}$ takes constant sections to constant sections, (3.6.18) generalizes to

$$\widetilde{B}(y, b_1 \wedge b_2 \wedge b_3) = (y, 1)$$

for every $y \in \widetilde{M}$. What is more, for every $x \in \widetilde{M}$ the exterior product $\mathrm{vol}(\underline{f_\Delta})_x$ is defined as the π_1-orbit of $(y, b_1 \wedge b_2 \wedge b_3)$, where y is any lift of x to $\widetilde{M}$. Q.E.D.

Corollary 3.6.11. *Assume that M is an open 3-manifold with toroidal ends admitting a complete hyperbolic structure of finite volume. Consider M either with the complete hyperbolic metric or with any incomplete hyperbolic metric resulting from a small deformation of the complete structure. Let $\mathfrak{A}$ be a geometric atlas satisfying the assumptions of Theorem 3.6.10. Then*

$$\mathbb{T}_{\mathrm{Ad}}(M, \gamma) = \mathbb{T}\big(\check{C}^\bullet(\mathfrak{U}(\mathfrak{A}), \mathscr{K}), \mathfrak{A}(\underline{b}), \underline{h}\big),$$

where $\underline{b}$ is an arbitrarily fixed basis of $\mathfrak{sl}_2\mathbb{C}$ and $\underline{h}$ is a cohomology basis balanced with respect to γ.

Proof. Our assumptions imply that $\chi(M) = 0$. Thus, according to Corollary 3.6.8, the choice of the basis $\underline{b} \subset \mathfrak{sl}_2\mathbb{C}$ does not affect the value of the torsion. Q.E.D.

3.7 Cellular geometric bases revisited

Since Porti's original definition [46] of the non-abelian hyperbolic torsion (Definition 3.5.2) is stated entirely in terms of CW-complexes, we wish to briefly indicate how to generalize the results of the preceding section to the cellular setting. We shall follow Steenrod [49] in his approach to cellular cochain complexes with local coefficients.

As before, let M be a cusped hyperbolic 3-manifold. Suppose X is an abstract cell complex and that $X(M) \subset M$ is a geometric realization of X as a deformation retract of M.

We are going to endow the cellular cochain complex $C^\bullet(X; \mathscr{K})$ with a basis. We start by fixing a basis $\underline{b}$ of $\mathfrak{sl}_2\mathbb{C} \cong \mathscr{K}(\mathbb{H}^3)$. For every cell s of X, we choose a point $*(s) \in X(M)$ lying in the geometric realization of s. Suppose $\varphi_s \in \mathscr{D}_{*(s)}$ is an arbitrarily fixed germ of an orientation-preserving developing map at $*(s)$. We may now equip the germ space $\mathscr{K}_{*(s)}$ with the pullback basis $(\varphi_s)_*^{-1}(\underline{b})$. When the cell s is oriented, we may consider the cochains 'dual' to this pullback basis. These cochains evaluate to the elements of the pullback basis on the oriented cell s and to zero on all other cells. This construction can be carried out for all cells s of X, yielding a basis of $C^\bullet(X; \mathscr{K})$.

Definition 3.7.1. Any basis of $C^\bullet(X; \mathscr{K})$ constructed as above is called a Steenrod geometric basis.

By arguments similar to those in the preceding section, any Steenrod geometric basis of the cellular cochain complex $C^\bullet(X; \mathscr{K})$ gives the same value of the Reidemeister torsion as a geometric basis in the sense of Porti [46]. In particular, the choices of the points $*(s)$, of the germs φ_s and of the orientations of the cells do not affect this construction of the torsion.

Note that by working with the stalks of the sheaf $\mathscr{K}$, one in each cell, we have eliminated the need to consider the carrier sets U_Δ. For more details, we refer the reader to [49, Section II], where this approach to cellular cochains was first introduced.

3.7.1 The dual cell decomposition

An ideal triangulation $\mathcal{T}$ of M defines a cell decomposition of M called the *dual cell decomposition*. The dual decomposition is a 2-dimensional CW-complex which can be thought of as the dual spine of the triangulation. It can be constructed as follows:

- The 0-cells are in a bijective correspondence with the tetrahedra $\{\Delta_1, \dots, \Delta_N\}$ of $\mathcal{T}$.
- The 1-cells are in a bijective correspondence with the faces of $\mathcal{T}$. If F is a face of $\mathcal{T}$, denote by Δ and Δ' the tetrahedra on both sides of F. The 1-cell dual to F is attached to the 0-cells dual to Δ and Δ'.
- The 2-cells are in a bijective correspondence with the edges of $\mathcal{T}$. For an edge e_i, denote by $F_1, \dots, F_r$ the book of faces adjacent to e_i (up to a cyclic permutation). The attachment map of the 2-cell dual to e_i sends the circle (the boundary of a disc) to the cellular cycle formed by the 1-cells dual to the faces $F_1, \dots, F_r$ in the cyclic order in which they appear around e_i.

Suppose $\mathcal{T}$ is a geometric ideal triangulation. We can now easily construct a geometric realisation of the dual cell decomposition as a retract of M. To this end, we can place a single 0-cell in the interior of each tetrahedron of $\mathcal{T}$. The 1-cells will then form arcs connecting the neighbouring tetrahedra across their shared faces (note that the two tetrahedra touching along a face need not be distinct). Finally, the 2-cells are dual to the edges of the triangulation. One may realise them geometrically as surfaces transverse to the edges of $\mathcal{T}$ and bounded by the cycles of arcs crossing the faces – see Figure 3.7.1.

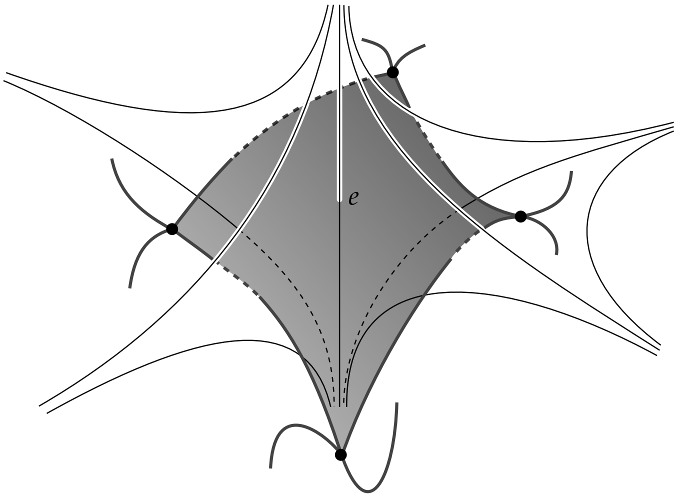


Figure 3.7.1: A fragment of the geometric realisation of the cell decomposition of M dual to an ideal triangulation. The shaded surface is a 2-cell transverse to the edge e; the thick lines are 1-cells, each of them dual to a face of the triangulation.

Remark 3.7.2. Epstein and Penner constructed similar CW-decompositions independently of ideal triangulations; in particular, on p. 77 of [15], they describe the *collision locus* of expanding horospherical hypersurfaces in a non-compact, complete, finite volume hyperbolic manifold of arbitrary dimension. In dimension 3, this set can be formally defined by fixing a collection of disjoint horospherical tori about all cusps of M and taking the corresponding *cut locus*, i.e., the set of points in M admitting more than one shortest path to the chosen tori. Alternatively, the cut locus can be understood as the quotient in M of a part of the boundary of a Ford fundamental domain $F \subset \mathbb{H}^3$ for M. This construction is dual to Epstein and Penner's ideal polyhedral decompositions of non-compact, complete, finite volume hyperbolic manifolds. At the same time, even just in the 3-dimensional case, it is not known [27] whether the polyhedra of these decompositions can always be subdivided into positive volume ideal tetrahedra (if they can, the cut locus becomes a concrete realisation in M of the dual cell decomposition of the resulting geometric ideal triangulation). It may therefore be argued that the CW-decomposition given by the cut locus is a more fundamental concept than a geometric ideal triangulation.

Chapter 4

Cohomological interpretation of Thurston's gluing equations

Throughout this chapter, we are assuming that M is an open 3-manifold admitting a complete hyperbolic structure of finite volume and that $\mathcal{T}$ is a geometric ideal triangulation of M with N tetrahedra. As described by Thurston [50], the complete hyperbolic structure on M can be deformed around each cusp into incomplete hyperbolic structures. These incomplete structures will be of essential importance throughout the chapter.

Our present goal is to establish a cohomological interpretation of Thurston's hyperbolicity equations. Recall from Section 1.2 that there are two kinds of gluing equations: the consistency conditions around the edges of $\mathcal{T}$ and the completeness conditions at the toroidal ends of M.

In Section 4.3 below, we are going to see that the holomorphic derivatives of the edge consistency conditions describe the essential part of the connecting homomorphism in a certain cohomology long exact sequence with coefficients in the sheaf of germs of Killing fields. This result appears to be new and is stated as Theorem 4.3.1.

Subsequently, we turn to the study of the completeness condition. We prove that the calculation of the derivatives of the log-parameters along peripheral curves in M is essentially equivalent to taking the cup product with the Poincaré duals of these curves in cohomology with coefficients in $\mathscr{K}$. This result, stated as Theorem 4.4.3 below, is strongly inspired by the work of Futer and Guéritaud [19], which in turn is partially based on W. Neumann's [41] study on the combinatorics of ideal triangulations. More precisely, Theorem 4.4.3 can be viewed as a complexified version of Lemma 4.4 in [19].

An interesting observation of Neumann and Zagier [40] is that the gluing equation matrices G, G', G'' possess certain symplectic properties. These properties were studied in more detail in Neumann's paper [41], where in Lemma 4.3–(iii) it is stated that for two primitive elements $\alpha, \beta \in H_1(\partial\bar{M}; \mathbb{Z})$, the intersection pairing $\iota(\alpha, \beta)$ can be computed in terms of the gluing matrices $G^{\square}_{\alpha}$, $G^{\square}_{\beta}$, which in this case are simply row vectors in $\mathbb{Z}^N$ (cf. (1.3.11)). Since the Poincaré duality takes the intersection pairing into the cup product, we only need to adapt Neumann's result to work with coefficients in a restriction of the sheaf $\mathscr{K}$ to a neighbourhood of the ends of M. As an application of Theorem 4.4.3, we prove that we can obtain the correct normalization of the non-abelian Reidemeister torsion directly from the combinatorics of the gluing equations as studied by Choi [9].

4.1 A cohomology long exact sequence associated to an ideal triangulation

In this section, we allow the manifold M to have any number $k > 0$ of toroidal ends. We assume M is equipped with a hyperbolic structure of finite volume which is either the complete structure or a small deformation of it. We denote by $\mathscr{K}$ the sheaf of germs of Killing vector fields on M. As in Section 1.2, we denote the edges of $\mathcal{T}$ by $e_1, \dots, e_N$.

Each edge e_i is a closed subset of M, disjoint from all other edges. Let $\{\nu_i\}_{i=1}^N$ be a collection of pairwise disjoint open neighbourhoods of the edges, i.e., $e_i \subset \nu_i$ for all i and $\nu_i \cap \nu_j = \emptyset$ whenever $i \neq j$.

We define $M_0 = M \setminus \bigcup_{i=1}^N \nu_i$. Since M_0 is a closed subspace of M, it determines a short exact sequence of sheaves on M,

$$0 \to \mathscr{K}_{M,M_0} \to \mathscr{K} \to \mathscr{K}_{M_0} \to 0. \tag{4.1.1}$$

The main object of our study in this section is the cohomology long exact sequence corresponding to the short exact sequence (4.1.1), which we can write as

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(M;\mathscr{K}_{M,M_0}) & \longrightarrow & H^0(M;\mathscr{K}) & \longrightarrow & H^0(M;\mathscr{K}_{M_0}) & \\ & H^1(M;\mathscr{K}_{M,M_0}) & \longrightarrow & H^1(M;\mathscr{K}) & \longrightarrow & H^1(M;\mathscr{K}_{M_0}) & \\ & H^2(M;\mathscr{K}_{M,M_0}) & \longrightarrow & H^2(M;\mathscr{K}) & \longrightarrow & H^2(M;\mathscr{K}_{M_0}) & \longrightarrow 0, \end{array} \tag{4.1.2}$$

terminating after cohomology groups in degree two. In fact, the following proposition implies that the last term $H^2(M;\mathscr{K}_{M_0})$ vanishes as well.

Proposition 4.1.1. *M_0 has the homotopy type of a graph.*

Proof. Up to homotopy, M_0 can be viewed as a union of *doubly truncated tetrahedra*. As seen in Figure 4.1.1, a doubly truncated tetrahedron can be deformation-rectracted onto its spine, a four-legged graph called a *tetrapod*. These deformations can be performed in each doubly truncated tetrahedron of M_0, showing that M_0 deformation-retracts onto the graph given by the union of the tetrapods. Q.E.D.

4.1.1 Cohomology vanishing results

We are now going to establish the vanishing of certain terms ocurring in (4.1.2). To start with, consider the exact sequence

$$0 \to H^0(M;\mathscr{K}_{M,M_0}) \to H^0(M;\mathscr{K}).$$

The rightmost term is zero by Lemma 3.4.1. This shows that $H^0(M;\mathscr{K}_{M,M_0}) = 0$ as well.

Next, we calculate the cohomology group $H^1(M;\mathscr{K}_{M,M_0})$. The sheaf $\mathscr{K}_{M,M_0}$ vanishes on M_0, i.e., $\mathscr{K}_{M,M_0}(U) = 0$ whenever $U \cap M_0 \neq \emptyset$. Note that the cohomology groups of M with coefficients in $\mathscr{K}_{M,M_0}$ are the same as the relative cohomology groups of the pair (M, M_0) with coefficients in $\mathscr{K}$. For each $i \in \{1, \dots, N\}$, let $\nu_i' \subset \nu_i$ be a smaller tubular neighbourhood of

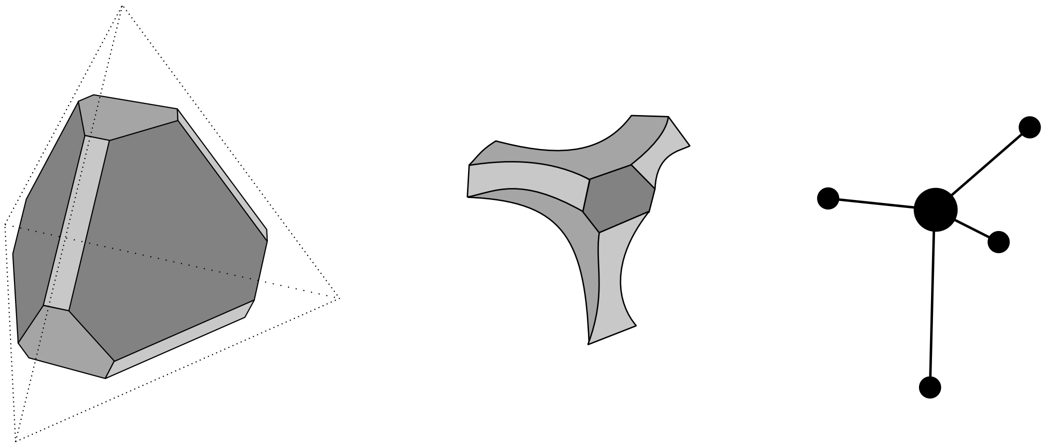


Figure 4.1.1: LEFT: A *doubly truncated tetrahedron*, formed by removing neighbourhoods of the long edges of a truncated tetrahedron (cf. Figure 1.2.2, left). MIDDLE: We can shrink a doubly truncated tetrahedron by excavating solid material near the truncated vertices and edges, but leaving central parts of the big hexagonal faces (dark grey) intact. RIGHT: A tetrapod onto which the doubly truncated tetrahedron deformation-retracts. One may think of the big dot as of a point in the centre of the tetrahedron, while the smaller dots are placed at the orthocentres of the faces.

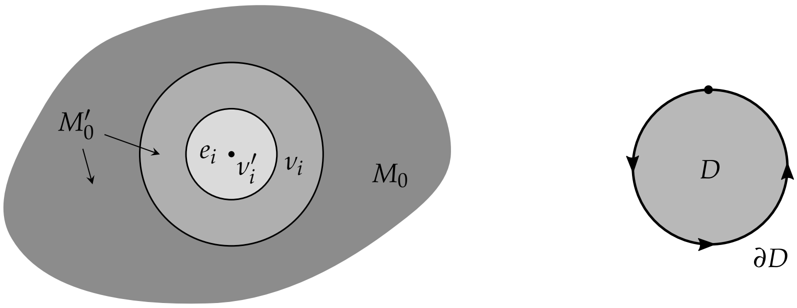


Figure 4.1.2: LEFT: A cross-sectional view of concentric tubular neighbourhoods of an edge e_i. The edge is perpendicular to the plane of the figure. M_0 is a deformation retract of M'_0. RIGHT: A CW-decomposition of a disc D with one 0-cell, one 1-cell and one 2-cell, with ∂D as the 1-skeleton.

the edge e_i (for instance, we can take ν'_i to have half the radius of ν_i). Define $M'_0 = M \setminus \bigcup_{i=1}^N \nu'_i$ (see left panel of Figure 4.1.2). Since M_0 is a deformation retract of M'_0, the relative cohomology of the pair (M, M'_0) is the same as that of (M, M_0). But excision implies that the relative cohomology of the pair (M, M'_0) coincides with the relative cohomology of $(M \setminus M_0, M'_0 \setminus M_0) = (\bigcup_{i=1}^N \nu_i, \bigcup_{i=1}^N \nu_i \setminus \nu'_i) = \bigsqcup_{i=1}^N (\nu_i, \nu_i \setminus \nu'_i)$. A projection of these tubular neighbourhoods onto a cross-sectional plane as in Figure 4.1.2, left, is a homotopy equivalence. It maps ν_i onto the larger disc and ν'_i onto the smaller, concentric disc. Hence, it suffices to understand $H^1(D, \partial D; \mathscr{K})$ where D is the cross-sectional disc of ν'_i. Since a disc can be given a CW-decomposition as shown on the right panel of Figure 4.1.2, with no 1-cells in its interior $D \setminus \partial D$, we conclude that $H^1(D, \partial D; \mathscr{K}) = 0$. Therefore, the cohomology group $H^1(M; \mathscr{K}_{M,M_0})$ vanishes. We summarize these considerations in the proposition below.

Proposition 4.1.2. *The following cohomology groups occurring in* (4.1.2) *vanish:*

$$H^0(M; \mathscr{K}_{M,M_0}), \;\; H^0(M; \mathscr{K}), \;\; H^0(M; \mathscr{K}_{M_0}), \;\; H^1(M; \mathscr{K}_{M,M_0}), \;\; H^2(M; \mathscr{K}_{M_0}). \tag{4.1.3}$$

Proof. The vanishing of $H^2(M; \mathscr{K}_{M_0})$ is a direct corollary of Proposition 4.1.1. The foregoing reasoning establishes the vanishing of all remaining cohomology groups in (4.1.3) except for $H^0(M;\mathscr{K}_{M_0})$. This, however, easily follows from the exactness of the sequence

$$H^0(M;\mathscr{K}) \to H^0(M;\mathscr{K}_{M_0}) \to H^1(M;\mathscr{K}_{M,M_0}),$$

in which the rightmost and leftmost terms were already shown to vanish. Q.E.D.

The above proposition implies that the long exact sequence (4.1.2) reduces to at most four nonzero terms, taking the form

$$0 \to H^1(M;\mathscr{K}) \to H^1(M;\mathscr{K}_{M_0}) \xrightarrow{\partial} H^2(M;\mathscr{K}_{M,M_0}) \to H^2(M;\mathscr{K}) \to 0. \tag{4.1.4}$$

We are going to see that for every hyperbolic 3-manifold M with $k > 0$ toroidal ends, all of these four terms are non-trivial. The exactness of (4.1.4) implies that the essential information about this sequence is encoded in the connecting homomorphism ∂, while the leftmost and rightmost terms can be identified with its kernel and cokernel, respectively.

4.1.2 The second cohomology group of the relative subsheaf

The excision argument illustrated in Figure 4.1.2 and used in the lead-up to Proposition 4.1.3 can also help us understand the cohomology group $H^2(M;\mathscr{K}_{M,M_0})$. Since the neighbourhoods ν_i are assumed disjoint, we can write

$$H^2(M;\mathscr{K}_{M,M_0}) \cong \prod_{i=1}^{N} H^2(M;\mathscr{K}_{\overline{\nu_i},\partial\nu_i}).$$

Therefore, it suffices to understand the cohomology group $H^2(M;\mathscr{K}_{\overline{\nu_i},\partial\nu_i})$ for each i.

Proposition 4.1.3. *For every $i \in \{1,\dots,N\}$, there is an isomorphism*

$$\sigma_i\colon \mathscr{K}(\nu_i) \xrightarrow{\cong} H^2(M;\mathscr{K}_{\overline{\nu_i},\partial\nu_i}).$$

As a result, $H^2(M;\mathscr{K}_{M,M_0}) \cong \prod_{i=1}^{N} \mathscr{K}(\nu_i)$.

Proof. To make the proof completely elementary, we can use the cellular decomposition of the disc D shown on the right of Figure 4.1.2 and restrict the sheaf $\mathscr{K}$ to D, embedded as a disc transversal to the edge e_i with $\partial D \subset \partial M_0$. Since the only 1-cell of this CW-complex is fully contained in ∂D, the cellular coboundary map $\delta^1\colon C^1(D,\partial D;\mathscr{K}_D) \to C^2(D,\partial D;\mathscr{K}_D)$ is zero. Hence, the entire cochain space $C^2(D,\partial D;\mathscr{K}_D)$ is canonically isomorphic to $H^2(D,\partial D;\mathscr{K}_D)$. Since the 2-cell forming the interior of D is a deformation retract of ν_i, we obtain an isomorphism $\sigma_i\colon \mathscr{K}(\nu_i) \to H^2(\overline{\nu_i},\partial\nu_i;\mathscr{K}) = H^2(M;\mathscr{K}_{\overline{\nu_i},\partial\nu_i})$. Note that σ_i depends on the chosen orientation of the disc D above; reversing the orientation replaces σ_i with its negative. Q.E.D.

Remark 4.1.4. As seen in the above proof, the sign of σ_i depends on the chosen orientation of the disc D. A more elegant way to fix this indeterminacy is perhaps by orienting the edge e_i of the triangulation $\mathcal{T}$. Since M is oriented by the volume form vol, there is a unique orientation O of D which satisfies $\mathrm{vol} = O \wedge E$, where E is the chosen orientation of e_i. This allows us to associate the map σ_i to an oriented edge (e_i, E), with $(e_i, -E)$ corresponding to $-\sigma_i$.

4.2 An algebraic perspective on the gluing equations

In this section, we interpret the relationship between M and M_0 algebraically, by looking at the fundamental groups of these spaces. This interpretation is well known and is in fact mentioned in Thurston's lecture notes [50]; we include it here in order to put our analysis of the gluing equations in a broader context.

We start by briefly discussing the standard method of constructing representations of finitely presented groups and then apply it to our situation.

Suppose Γ is a finitely presented group with a presentation

$$\Gamma = \langle g_1, g_2, \ldots, g_r \mid R_1, R_2, \ldots, R_s \rangle.$$

In other words, Γ is given as a quotient of the free group $F_r := \langle g_1, g_2, \ldots, g_r \rangle$ by the subgroup N_R normally generated by all relators R_i, $1 \leq i \leq s$; these groups fit into the short exact sequence

$$0 \to N_R \to F_r \to \Gamma \to 0. \tag{4.2.1}$$

Given the task of describing the set $\mathrm{Hom}(\Gamma, G)$ of all representations of Γ in a fixed group G, we may start by looking at $\mathrm{Hom}(F_r, G)$. Since F_r is free, any function $f\colon \{g_1, \ldots, g_r\} \to G$ defines a representation of F_r. Hence, $\mathrm{Hom}(F_r, G) = G^r$. Moreover, a representation of F_r in G descends to the quotient $\Gamma = F_r / N_R$ if and only if it contains N_R in its kernel. Therefore, $\mathrm{Hom}(\Gamma, G)$ is the subset of G^r consisting of precisely those r-tuples of elements of G which vanish on the relators. More precisely, suppose the ith relator has the form

$$R_i = g_{a_1}^{\varepsilon_1} g_{a_2}^{\varepsilon_2} \cdots g_{a_l}^{\varepsilon_l},$$

with $a_1, \ldots, a_l \in \{1, \ldots, r\}$, $\varepsilon_1, \ldots, \varepsilon_l \in \mathbb{Z}$, $l \in \mathbb{N}$. Given an element $f \in \mathrm{Hom}(F_r, G)$, which we treat here as a function $f\colon \{g_1, \ldots, g_r\} \to G$, we say that f *vanishes on* R_i if

$$f(g_{a_1})^{\varepsilon_1} f(g_{a_2})^{\varepsilon_2} \cdots f(g_{a_l})^{\varepsilon_l} = 1 \in G. \tag{4.2.2}$$

In this way, the set $\mathrm{Hom}(\Gamma, G)$ is cut out from $\mathrm{Hom}(F_r, G) = G^r$ by the equations (4.2.2) for all $i \in \{1, \ldots s\}$.

Let us now restrict our attention to $\Gamma = \pi_1(M, *)$ and $G = PSL_2\mathbb{C}$. The inclusion $M_0 \hookrightarrow M$ induces a homomorphism $q\colon \pi_1(M_0, *) \to \pi_1(M, *)$. It is easy to see that q is surjective, since every path in M can be homotoped so that it misses the removed edge neighbourhoods, and hence can be assumed entirely contained in M_0. In other words, there is a short exact sequence

$$0 \to \mathrm{Ker}\, q \to \pi_1(M_0, *) \xrightarrow{q} \pi_1(M, *) \to 0. \tag{4.2.3}$$

Proposition 4.1.1 implies that the fundamental group of M_0 is free. Therefore, the exact sequence (4.2.3) is an instance of (4.2.1) with $\Gamma = \pi_1(M, *)$.

In order to make use of this analogy in the study of representations of $\pi_1(M, *)$, we should first look at the representations of the free group $\pi_1(M_0, *)$ and then find the subspace cut out by the suitable 'relator equations'. Thurston's gluing equations play an essential role in this description.

The proposition below is well-known; see for instance Section 4.2 of [50].

Proposition 4.2.1 (W. Thurston)**.** *Any assignment of shape parameters $(z_1, \ldots, z_N) \in P^N$ to the tetrahedra of $\mathcal{T}$ defines a hyperbolic structure on M_0. This structure extends to M if and only if $(z_1, \ldots, z_N) \in \mathcal{V}^+_{\mathcal{T}}$.*

In practical terms, the above proposition says that, as long as we are only interested in hyperbolic structures on M_0, we have complete freedom in assigning shape parameters to the tetrahedra with removed edges which make up M_0. Edge constraint equations only need to be imposed when the removed edge neighbourhoods are filled back in to form M.

As usual, we denote by N the number of tetrahedra in the triangulation $\mathcal{T}$. To calculate the rank of the free group $\pi_1(M_0)$, observe that

$$\operatorname{rank} H_0(M_0; \mathbb{Z}) - \operatorname{rank} H_1(M_0; \mathbb{Z}) = \chi(M_0) = \chi(M) - N = -N,$$

since M_0 is the result of removing N contractible subspaces from M. Because M_0 is connected, we know that $\operatorname{rank} H_0(M_0; \mathbb{Z}) = 1$. Therefore, $\operatorname{rank} H_1(M_0; \mathbb{Z}) = N + 1$. Since $H_1(M_0; \mathbb{Z})$ is the abelianization of the free group $\pi_1(M_0)$, we conclude that $\pi_1(M_0)$ is isomorphic to F_{N+1}, the free group on $N + 1$ generators.

To complete the analogy between $PSL_2\mathbb{C}$-representations of the free group $\pi_1(M_0, *)$ and the holonomy representations of $\pi_1(M_0, *)$ in $PSL_2\mathbb{C}$ defined by arbitrary shape parameters $(z_1, \ldots, z_N) \in P^N$, we need to pass to conjugacy classes of representations. This leads us to consider the $PSL_2\mathbb{C}$ *character variety* of F_{N+1}, which can be understood as a 'quotient' of the algebraic variety $\operatorname{Hom}(F_{N+1}, PSL_2\mathbb{C})$ by the action of $PSL_2\mathbb{C}$ by conjugation of representations.

More generally, the character variety $X(\Gamma)$ of a finitely presented group Γ in $PSL_2\mathbb{C}$ is the GIT quotient

$$X(\Gamma) = \operatorname{Hom}(\Gamma, PSL_2\mathbb{C}) \mathbin{/\!/} PSL_2\mathbb{C}.$$

We remark that character varietes are not manifolds in general, even in the case of free groups. We refer the reader to [16] for more details on singularities of character varietes and to [21] for an elementary definition of the GIT quotient operation // in the case of $PSL_2\mathbb{C}$.

For our purposes, we only need the following result concerning $PSL_2\mathbb{C}$ character varieties of free groups.

Proposition 4.2.2 (Heusener–Porti, [25, Proposition 5.8])**.** *When $N \geq 2$, the singular set of $X(F_{N+1})$ is precisely the set of* Ad*-reducible characters.*

An Ad-reducible character is a point in $X(F_{N+1})$ given by the conjugacy class of a representation $\varrho\colon F_{N+1} \to PSL_2\mathbb{C}$ whose adjoint $\operatorname{Ad}\varrho\colon F_{N+1} \to \operatorname{Aut}\mathfrak{sl}_2\mathbb{C}$ is a reducible representation. It turns out that Ad-reducible representations are the exception rather than the rule: by Lemma 3.13 in [25], the set of Ad-reducible representations is Zariski-closed in the variety $\operatorname{Hom}(F_{N+1}, PSL_2\mathbb{C})$. In other words, every point $[\varrho] \in X(F_{N+1})$ given by the conjugacy class of an Ad-*irr*educible representation ϱ has a Zariski-open neighbourhood consisting only of Ad-irreducible representations.

Corollary 4.2.3. *Let $[\varrho_*]$ be the conjugacy class of the holonomy representation of the hyperbolic structure on M_0 obtained by restricting the unique complete hyperbolic structure on M. Then $[\varrho_*]$ is a regular point of the character variety $X(\pi_1(M_0))$. Furthermore, $[\varrho_*]$ has an open neighbourhood consisting only of regular points.*

Proof. Since the adjoint of the holonomy representation of the complete hyperbolic structure on M is irreducible, the same is true of the representation of the free group $\pi_1(M_0)$ induced by the inclusion $M_0 \hookrightarrow M$ and its small deformations. Recall that N denotes the number of tetrahedra in the geometric ideal triangulation $\mathcal{T}$, which by Remark 1.2.6 must satisfy $N \geq 2$. Hence, Proposition 4.2.2 applies. Q.E.D.

For any collection of shape parameters $(z_1, \ldots, z_N) \in P^N$, denote by $E(z_1, \ldots, z_N)$ the point of the character variety $X(\pi_1(M_0))$ given as the conjugacy class of the holonomy representation of the hyperbolic structure on M_0 defined by $(z_1, \ldots, z_N)$. Note that $E(z_1, \ldots, z_N)$ is well-defined regardless of the choice of the basepoint, since we only consider the conjugacy classes of representations. We have therefore a well-defined map

$$E\colon P^N \to X(\pi_1(M_0)). \tag{4.2.4}$$

Denote by $z_* \in P^N$ the positively oriented solution of Thurston's gluing equations corresponding to the unique complete hyperbolic structure of finite volume on M. In the notation of Corollary 4.2.3, we have $E(z_*) = [\varrho_*]$. Let A be a Zariski-open neighbourhood of $[\varrho_*]$ in $X(\pi_1(M_0))$ consisting only of regular points. Since both P^N and A are analytic manifolds, it makes sense to ask whether the map E is analytic.

Theorem 4.2.4. *The map E of* (4.2.4) *is complex analytic on an open neighbourhood of z_*.*

Proof. Corollary 2.3.10 states that the holonomy representation $\mathrm{hol} = \varrho$ depends analytically on the shape parameters. Since the conjugacy classes of the holonomy representations of the complete hyperbolic structure, as well as its small deformations, are regular points of $X(\pi_1(M_0))$, the GIT quotient map $\mathrm{Hom}(\Gamma, PSL_2\mathbb{C}) \xrightarrow{//} X(\pi_1(M_0))$ is analytic at ϱ. Therefore, E is a composition of two analytic maps. Q.E.D.

The derivative of E at a point z can be understood algebraically. By the work of A. Weil [54], the tangent space to $X(\pi_1(M_0))$ at a regular point $[\varrho]$ is canonically isomorphic to the first cohomology group of M_0 with coefficients in the adjoint representation $\mathrm{Ad}\,\varrho$. On the other hand, Theorem 1.1.4 states that the local system defined by the representation $\mathrm{Ad}\,\varrho$ is none other than the sheaf of germs of Killing vector fields restricted to $M_0 \subset M$, with the geometry of given by gluing tetrahedra with shapes z. In this language, Weil's construction implies that the tangent space to $X(\pi_1(M_0))$ at the point $[\varrho]$ is isomorphic to the cohomology group $H^1(M; \mathscr{K}_{M_0})$. It is therefore possible to understand the derivative of E by considering instead the composition

$$T_z P^N \xrightarrow{D_z E} T_{[\varrho]} X(\pi_1(M_0)) \xrightarrow{\cong} H^1(M; \mathscr{K}_{M_0}), \tag{4.2.5}$$

where the point $z \in P^N$ defines the hyperbolic geometry on M_0 as well as the conjugacy class of its monodromy representation $[\varrho]$.

4.2.1 Complex lengths on the character variety

Following Thurston [50] and Neumann-Zagier [40], we may use the complex lengths of peripheral curves to understand a neighbourhood of $[\varrho_*]$ in the character variety $X(M) := X(\pi_1(M))$. Given a multicurve θ satisfying Assumption 1.2.1 and any $l \in \{1, \ldots, k\}$, we can conjugate a holonomy representation ϱ in such a way that

$$\varrho(\theta_l) = \pm \begin{bmatrix} e^{u_l/2} & * \\ 0 & e^{-u_l/2} \end{bmatrix}. \tag{4.2.6}$$

The number $u_l = u_l([\varrho])$, known as the complex length of θ_l, is then a multi-valued function on the character variety $X(M)$ satisfying

$$\operatorname{tr} \varrho(\theta_l) = \pm 2\cosh(u_l([\varrho])/2). \tag{4.2.7}$$

Note that the squares of the traces occurring on the left-hand side are coordinates on $X(M)$; cf. [25, 32].

Recall that in (1.3.5) we defined the map $y = y_\theta \colon U \to \mathcal{V}_{\mathcal{T}}^+$ which assigns to a log-parameter $u \in U$ the corresponding shape parameter solutions of the edge consistency equations. Since y has values in $\mathcal{V}_{\mathcal{T}}^+$, Proposition 4.2.1 allows us to regard the composition $E \circ y$ as taking values in $X(M)$. In other words, the map

$$E \circ y \colon U \to X(M) \tag{4.2.8}$$

takes a Dehn surgery parameter $u = (u_1, \ldots, u_k) \in U$ to the character $[\varrho]$ satisfying (4.2.7) for all l. Our convention that U is a neighbourhood of 0 provides a local choice of the branches of the logarithms of the eigenvalues in (4.2.6), but it does not resolve the sign indeterminacy. Thus, u and $-u$ define the same hyperbolic structure [50, 6, 40].

Definition 4.2.5.

(i) A finite-volume hyperbolic structure on an open 3-manifold M is called *totally incomplete* if it is incomplete at all ends of M.

(ii) A log-parameter $u \in U \subset \mathbb{C}^k$ is called *small* if $|u_l| < \pi$ for all l.

(iii) A hyperbolic structure on M is called a *small deformation* of the complete structure if it can be obtained by gluing positively oriented tetrahedra of $\mathcal{T}$ with shapes $z = y(u)$ where u is a small log-parameter.

Note that the notion of a small deformation depends on the choice of the multicurve θ used to define the log-parameters as well as on the chosen ideal triangulation; cf. also [9, p. 1360].

We observe that a hyperbolic structure with a small log-parameter u is totally incomplete if and only if $u_l \neq 0$ for all $l \in \{1, \ldots, k\}$. Therefore, the conjugacy classes of holonomy representations of totally incomplete structures form an open subset of $X(M)$ in the Euclidean topology. Moreover, by identifying the tangent space to $X(M)$ at a smooth point $[\varrho] \in (E \circ y)(U)$ with the first cohomology group of the sheaf $\mathscr{K}$ of infinitesimal isometries of the corresponding hyperbolic structure, we obtain the composition

$$T_u U \xrightarrow{D_u(E\circ y)} T_{[\varrho]} X(M) \xrightarrow{\cong} H^1(M; \mathscr{K}). \tag{4.2.9}$$

Since $\dim_{\mathbb{C}} H^1(M; \mathscr{K}) = k = \dim_{\mathbb{C}} U$, the composition of maps (4.2.9) is an isomorphism if and only if it is injective. Calculating the derivative of the squared trace of $\varrho(\theta_l)$, we obtain

$$\frac{d}{du_l} 4\cosh^2 \frac{u_l}{2} = 4\cosh\frac{u_l}{2} \sinh\frac{u_l}{2} = 2\sinh u_l.$$

Hence, $E \circ y$ is an analytic immersion whenever $u_l \notin \pi i \mathbb{Z}$ for all $l \in \{1, \ldots, k\}$.

Proposition 4.2.6. *Let u be a small log-parameter. Then the composition of maps in* (4.2.9) *is an isomorphism if and only if the hyperbolic structure on M corresponding to u is totally incomplete.*

Proof. The calculation above shows that the derivative of $E \circ y$ is an isomorphism if and only if $u_l \neq 0$ for all l, because u is a small log-parameter. This happens precisely at totally incomplete structures. Q.E.D.

4.3 Cohomological interpretation of the gluing equations

The goal of this section is to establish a fundamental relationship between the holomorphic derivatives of Thurston's gluing equations and the cohomology of the sheaf of germs of Killing vector fields. In what follows, the 3-manifold M is assumed to have a geometric ideal triangulation $\mathcal{T}$ with N tetrahedra and k toroidal ends. Furthermore, we fix a multicurve θ satisfying Assumption 1.2.1. The choice of θ allows us to write down the exact sequence $T\mathcal{GE}$ of (1.3.6). The following theorem identifies the exact sequence $T\mathcal{GE}$ as a subcomplex of the cohomology long exact sequence (4.1.4).

Theorem 4.3.1. *Let $M_0 = M_0(\mathcal{T})$ be the space defined in Section 4.1. Assume furthermore that the small log-parameter $u = (u_1, \ldots, u_k) \in U$ satisfies $u_l \neq 0$ for all l and consider M with the corresponding totally incomplete hyperbolic structure. Then there exists a commutative diagram with exact rows and columns*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_u U & \xrightarrow{Dy} & T_z P^N & \xrightarrow{Dg} & T_1 \mathbb{C}_\times^N & \xrightarrow{Dp} & \mathbb{C}^k & \longrightarrow & 0 \\
 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_3 & & \downarrow \eta_4 & & \\
0 & \longrightarrow & H^1(M;\mathscr{K}) & \longrightarrow & H^1(M;\mathscr{K}_{M_0}) & \xrightarrow{\partial} & H^2(M;\mathscr{K}_{M,M_0}) & \longrightarrow & H^2(M;\mathscr{K}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \operatorname{Coker} \eta_2 & \xrightarrow{\cong} & \operatorname{Coker} \eta_3 & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

in which the second row is (1.3.6) *and the third row is* (4.1.4)*. In particular, the maps η_1 and η_4 are isomorphisms.*

Proof. We split the proof into several steps, starting with the definitions of the maps $\eta_r, r < 4$.

Step 1: Definitions of η_1 and η_2

We define η_1 to be the composition of maps in (4.2.9). Since u is a small log-parameter, Proposition 4.2.6 guarantees that η_1 is an isomorphism.

To define η_2, recall that Proposition 4.2.1 states that any collection of shape parameters $(z_1, \ldots, z_N) \in P^N$ defines a hyperbolic structure on the manifold M_0. Hence, we may define η_2 as the composition of maps occurring in (4.2.5).

Step 2: Definition of η_3

For every i, we have the basis vector $\frac{\partial}{\partial x_i}$ of $T_1\mathbb{C}_\times^N$ corresponding to the edge e_i (see Section 1.3.1). Orient the edge e_i in an arbitrary way. Now the oriented edge e_i is an infinite oriented geodesic in M, so by virtue of Proposition 1.1.5, there exists a well-defined local Killing field $\mathfrak{h}_{e_i}$ on a neighbourhood of e_i. Continuing the field if necessary, we can take that neighbourhood to be exactly the tubular neighbourhood ν_i missing from M_0. Recall that $\mathfrak{h}_{e_i}$ is an infinitesimal translation in the direction of the orientation of e_i and its complex multiples are infinitesimal corkscrew motions about e_i. We define

$$\eta_3\colon T_1\mathbb{C}_\times^N \to H^2(M;\mathscr{K}_{M,M_0}), \quad \eta_3\Big(\frac{\partial}{\partial x_i}\Big) = \sigma_i(\mathfrak{h}_{e_i}),$$

where σ_i was constructed in Proposition 4.1.3 and the sign of σ_i is determined by the chosen orientation of e_i as in Remark 4.1.4. With this convention, η_3 is well defined, since changing the orientation of e_i simultaneously changes the signs of σ_i and of $\mathfrak{h}_{e_i}$. Thus, η_3 maps $\frac{\partial}{\partial x_i}$ to the cohomology class determined by infinitesimal unit speed translations along the edge e_i.

With the definitions of the maps η_2 and η_3 in place, their cokernels occurring in the commutative diagram are now defined as well.

Step 3: Description of the connecting homomorphism ∂

For every oriented edge e_i, the boundary of its open neighbourhood ν_i is a cylinder, so it retracts onto a circle. Using the ambient orientation of M, this circle can be oriented consistently with e_i using the right-hand rule detailed in Remark 4.1.4. Let ε_i denote the resulting oriented loop in M_0, considered up to homotopy; see Figure 4.3.1, Left.

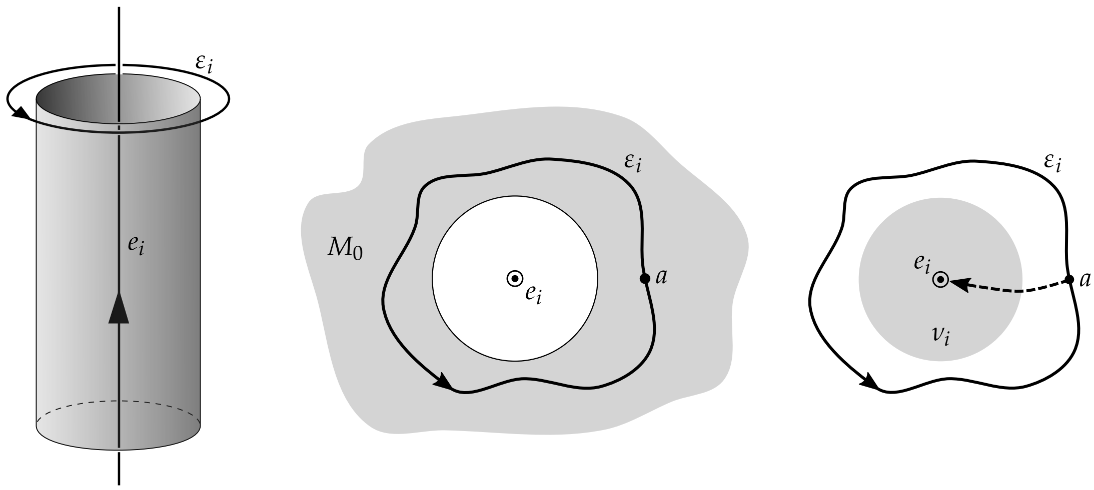


Figure 4.3.1: LEFT: Since M is oriented, for each oriented edge e_i, we can consider an oriented loop $\varepsilon_i \subset M_0$ encircling e_i in the positive direction (given by the right-hand rule). MIDDLE: View from top, along the edge e_i, with a starting point a chosen arbitrarily. RIGHT: A germ of a Killing field at a can be unambiguously continued onto the open neighbourhood ν_i of e_i.

Note that ε_i bounds a disc $D_i \subset M$, whence the local system $\mathscr{K}$ is trivial on ε_i. Given a cohomology class $\alpha \in H^1(M;\mathscr{K}_{M_0})$, we can therefore evaluate α on ε_i. Fixing a basepoint a on ε_i, we denote by $\mathrm{ev}_{\varepsilon_i}\colon H^1(M;\mathscr{K}_{M_0}) \to \mathscr{K}_a$ the evaluation map. The resulting germ of a Killing vector field $\mathrm{ev}_{\varepsilon_i}(\alpha) \in \mathscr{K}_a$ can be continued onto the disc $D_i \subset M$ bounded by ε_i, hence onto the entire open set ν_i (see Figure 4.3.1, Right). The triviality of $\mathscr{K}_{D_i}$ implies that the analytic continuation onto v_i is unique and defines a map

$$w_i\colon \mathcal{K}_a \to \mathcal{K}(v_i).$$

Moreover, the composition $w_i \circ \mathrm{ev}_{\varepsilon_i}$ does not depend on the choice of the starting point a.

Suppose the disc D_i is oriented in such a way that its oriented boundary is exactly ε_i. Then the map σ_i of Proposition 4.1.3 is unambiguously defined. It is now easy to see that locally, in a neighbourhood of e_i, the connecting homomorphism $\partial\colon H^1(M;\mathcal{K}_{M_0}) \to H^2(M;\mathcal{K}_{M,M_0})$ is given by the composition $\sigma_i \circ w_i \circ \mathrm{ev}_{\varepsilon_i}$. Using the second part of Proposition 4.1.3, we obtain

$$\partial = \sum_{i=1}^{N} (\sigma_i \circ w_i \circ \mathrm{ev}_{\varepsilon_i}).$$

Step 4: Commutativity of the connecting square

We are now going to prove the commutativity of the square

$$\begin{array}{ccc} T_zP^N & \xrightarrow{\quad Dg \quad} & T_1\mathbb{C}^N_\times \\ \Big\downarrow{\scriptstyle \eta_2} & & \Big\downarrow{\scriptstyle \eta_3} \\ H^1(M;\mathcal{K}_{M_0}) & \xrightarrow{\ \partial\ } & H^2(M;\mathcal{K}_{M,M_0}). \end{array} \tag{4.3.1}$$

Let $i \in \{1,\dots,N\}$ be an arbitrarily chosen index and e_i the corresponding edge of $\mathcal{T}$, equipped with an arbitrary orientation. Similarly, consider the jth tetrahedron Δ_j of $\mathcal{T}$ and the associated basis element $\frac{\partial}{\partial z_j} \in T_zP^N$. To simplify the discussion, we can choose a geometric chart on v_i which sends the oriented edge e_i to the oriented geodesic $(0,\infty)$ in the upper-halfspace model of $\mathbb{H}^3$. In this local chart, the holonomy of the hyperbolic structure on M_0 along the curve ε_i is a composition of complex homotheties given by the shape parameters labeling edges of the tetrahedra incident to e_i. In other words,

$$\mathrm{hol}_z\,\varepsilon_i = (x \mapsto g_i(z)x) \in PSL_2\mathbb{C}, \tag{4.3.2}$$

where $z = (z_1,\dots z_N) \in P^N$ and g_i is the ith component of the gluing map g defined in (1.2.8).

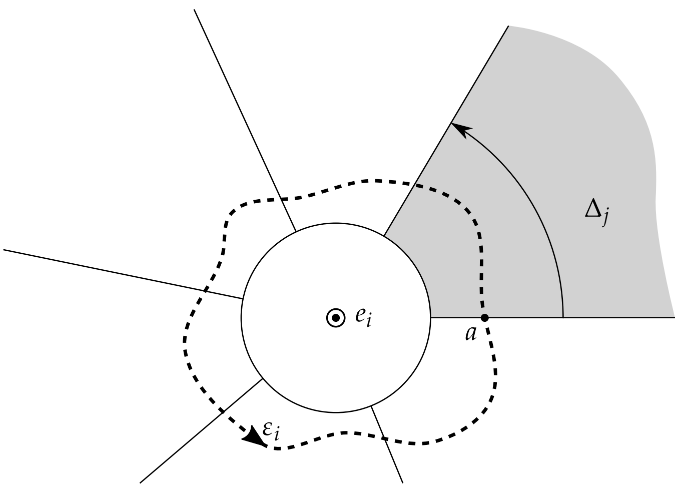


Figure 4.3.2: When the oriented edge e_i is lifted to the oriented geodesic $(0,\infty) \subset \mathbb{H}^3$, the holonomy along the curve ε_i becomes a composition of complex homotheties whose dilation factors are the shape parameters associated to the edges of the tetrahedra of $\mathcal{T}$ incident to e_i. One such tetrahedron Δ_j is highlighted. In general, a tetrahedron may have several of its edges glued to e_i.

In order to compute the ith component of $\partial \circ \eta_2(\frac{\partial}{\partial z_j})$, we need to evaluate on ε_i the cohomology class in $H^1(M; \mathscr{K}_{M_0})$ resulting from differentiating shape parameters with respect to z_j. Using Weil's method [54], this amounts to calculating

$$\frac{\partial}{\partial s_j}\bigg|_{s=z} \mathrm{hol}_s\, \varepsilon_i\; (\mathrm{hol}_z\, \varepsilon_i)^{-1} = \frac{\partial}{\partial s_j}\bigg|_{s=z} \begin{bmatrix} \sqrt{g_i(s)} & 0 \\ 0 & 1/\sqrt{g_i(s)} \end{bmatrix},$$

where we have written the homothety (4.3.2) as an $SL_2\mathbb{C}$ matrix defined only up to sign, cf. (2.3.2). Note that we were able to omit the term $(\mathrm{hol}_z\, \varepsilon_i)^{-1}$, since the holonomy at the point $z \in \mathcal{V}_{\mathcal{T}}^+$ is trivial. Using the notation of (1.2.5), we define an analytic function

$$\ell \colon P^N \to \mathbb{C}^N, \quad \ell(z) = GZ + G'Z' + G''Z'' - (2\pi\sqrt{-1}, \dots, 2\pi\sqrt{-1})^\top, \tag{4.3.3}$$

where the vector with entries $2\pi\sqrt{-1}$ is designed to ensure that $\ell|_{\mathcal{V}_{\mathcal{T}}^+} \equiv 0$ (cf. equation (1.2.6)). Denoting by $\ell_i \colon P^N \to \mathbb{C}$ the ith component of ℓ, we have $g_i(z) = \exp(\ell_i(z))$. Hence,

$$\frac{\partial}{\partial s_j}\bigg|_{s=z} \begin{bmatrix} \sqrt{g_i(s)} & 0 \\ 0 & 1/\sqrt{g_i(s)} \end{bmatrix} = \frac{\partial \ell_i(s)}{\partial s_j}\bigg|_{s=z} \frac{\partial}{\partial t}\bigg|_{t=0} \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} = \frac{\partial \ell_i(z)}{\partial z_j}\mathfrak{h}, \tag{4.3.4}$$

where the last equality follows from (1.1.5). On the other hand, since $\ell_i(z)$ differs from the left-hand side of (1.2.4) by a constant, we obtain (regardless of the choice of branches of the logarithms)

$$\frac{\partial}{\partial z_j}\ell_i(z) = G_{ij}\frac{\partial \log z_j}{\partial z_j} + G'_{ij}\frac{\partial \log z'_j}{\partial z_j} + G''_{ij}\frac{\partial \log z''_j}{\partial z_j} = G_{ij}\zeta_j + G'_{ij}\zeta'_j + G''_{ij}\zeta''_j,$$

using notation of Convention 1.3.3. We have thus calculated

$$\frac{\partial}{\partial s_j}\bigg|_{s=z} \mathrm{hol}_s\, \varepsilon_i\; (\mathrm{hol}_z\, \varepsilon_i)^{-1} = \left(G_{ij}\zeta_j + G'_{ij}\zeta'_j + G''_{ij}\zeta''_j\right)\mathfrak{h}.$$

This result holds when the edge e_i is lifted to the oriented geodesic $(0, \infty) \subset \mathbb{H}^3$. According to Proposition 1.1.5, there is a well-defined pullback of $\mathfrak{h}$ to ν_i, denoted by $\mathfrak{h}_{e_i}$. Recall that $\mathfrak{h}_{e_i}$ is a unit-speed infinitesimal translation along the oriented edge e_i. In this way, we can write

$$\partial \circ \eta_2(\tfrac{\partial}{\partial z_j}) = \sum_{i=1}^{N} \sigma_i \left((G_{ij}\zeta_j + G'_{ij}\zeta'_j + G''_{ij}\zeta''_j)\mathfrak{h}_{e_i} \right).$$

As seen in Step 2, the right-hand side is well-defined since both σ_i and $\mathfrak{h}_{e_i}$ change sign when the orientation of e_i is changed.

We now consider $\eta_3 \circ Dg$. Recall that the Jacobian matrix of g with respect to the bases $\{\frac{\partial}{\partial z_j}\}_{j=1}^N \subset T_z P^N$, $\{\frac{\partial}{\partial x_i}\}_{i=1}^N \subset T_1\mathbb{C}_\times^N$ was given in (1.3.10). As a consequence,

$$\begin{aligned} \eta_3 \circ Dg(\tfrac{\partial}{\partial z_j}) &= \eta_3 \left(\sum_{i=1}^{N} (G_{ij}\zeta_j + G'_{ij}\zeta'_j + G''_{ij}\zeta''_j)\frac{\partial}{\partial x_i} \right) \\ &= \sum_{i=1}^{N} \left(G_{ij}\zeta_j + G'_{ij}\zeta'_j + G''_{ij}\zeta''_j\right)\sigma_i(\mathfrak{h}_{e_i}) = \partial \circ \eta_2(\tfrac{\partial}{\partial z_j}), \end{aligned}$$

proving that the square (4.3.1) commutes.

Step 5: Definition of η_4

The map η_4 can now be defined by a standard diagram chase argument, which we include here in the interest of completeness. We wish to set $\eta_4 = \iota^* \circ \eta_3 \circ (Dp)^{-1}$, where $\iota^* \colon H^2(M; \mathscr{K}_{M,M_0}) \to H^2(M; \mathscr{K})$ is the map from the cohomology long exact sequence. For η_4 to be well-defined, we need to ensure that $\iota^* \circ \eta_3$ sends $\operatorname{Ker} Dp$ to zero. This is immediate, since $(\iota^* \circ \eta_3)(\operatorname{Ker} Dp) = \iota^* (\operatorname{Im} \eta_3 \circ Dg) = \iota^* (\operatorname{Im} \partial \circ \eta_2) \subseteq \operatorname{Im} \iota^* \circ \partial = \{0\}$.

Step 6: Injectivity of $\eta_\bullet$

At present, all maps and spaces ocurring in the diagram are well-defined and all squares in the top part have been shown to commute. The injectivity of η_1 was observed in Step 1. It remains to prove that η_r is injective for all $r > 1$.

Consider first $\eta_3 \colon T_1\mathbb{C}^N_\times \to H^2(M; \mathscr{K}_{M,M_0})$ and recall that Proposition 1.1.5 allows us to write $H^2(M; \mathscr{K}_{M,M_0}) = \prod_i \sigma_i\,(\mathscr{K}(\nu_i))$. Since for every $i \in \{1, \dots, N\}$ the basis element $\frac{\partial}{\partial x_i} \in T_1\mathbb{C}^N_\times$ is sent by η_3 to a nonzero element of the ith factor of the above product, it follows that η_3 is injective. The injectivity of η_2 now follows effortlessly from an application of the Four Lemma to the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_u U & \xrightarrow{\;Dy\;} & T_z P^N & \xrightarrow{\;Dg\;} & T_1\mathbb{C}^N_\times \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} \\
0 & \longrightarrow & H^1(M; \mathscr{K}) & \longrightarrow & H^1(M; \mathscr{K}_{M_0}) & \xrightarrow{\;\partial\;} & H^2(M; \mathscr{K}_{M,M_0})
\end{array}$$

in which η_1 and η_3 have already been shown injective.

Consider the map $[\partial] \colon \operatorname{Coker} \eta_2 \to \operatorname{Coker} \eta_3$ induced by ∂ on cokernels. Since η_1 is surjective, $\operatorname{Coker} \eta_1 = 0$ and consequently $[\partial]$ is injective. Furthermore,

$$\dim H^1(M; \mathscr{K}_{M_0}) = \dim H^2(M; \mathscr{K}_{M,M_0}) = 3N$$

and $\dim T_z P^N = \dim T_1\mathbb{C}^N_\times = N$, so the cokernels of η_2 are η_3 have equal dimension $2N$. Therefore, the cokernels of η_2 and η_3 are isomorphic under $[\partial]$.

Finally, consider the commutative diagram

$$\begin{array}{ccccccc}
 & & T_1\mathbb{C}^N_\times & \xrightarrow{\;Dp\;} & \mathbb{C}^k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_3} & & \downarrow{\scriptstyle \eta_4} & & \\
H^1(M; \mathscr{K}_{M_0}) & \xrightarrow{\;\partial\;} & H^2(M; \mathscr{K}_{M,M_0}) & \longrightarrow & H^2(M; \mathscr{K}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\operatorname{Coker} \eta_2 & \xrightarrow{\;\cong\;} & \operatorname{Coker} \eta_3 & \longrightarrow & \operatorname{Coker} \eta_4 & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0, & & & &
\end{array}$$

in which the top two rows are exact. By the Eight Lemma, the bottom row is also exact, implying $\operatorname{Coker} \eta_4 = 0$. This shows that η_4 is surjective. The injectivity of η_4 now follows from dimension counting: by Lemma 3.4.1, we have $\dim H^2(M; \mathscr{K}) = k$. Q.E.D.

We remark that the above result cannot be extended to the case when any of the ends of M are complete, since then the corresponding component of the map η_1 would become zero, as seen from Proposition 4.2.6. Nonetheless, the commutativity of the connecting square shown in Step 5 of the proof holds even in the complete case.

4.4 Cohomological interpretation of complex lengths

In this section, we explain how to think of the derivatives of the log-parameters $u = (u_1, \dots, u_k)$ in cohomological terms. Recall that, having fixed a multicurve $\theta = \{\theta_1, \dots, \theta_k\}$ satisfying Assumption 1.2.1, we have, for every $l \in \{1, \dots, k\}$,

$$u_l = G_{\theta_l,j} \log z_j + G'_{\theta_l,j} \log z'_j + G''_{\theta_l,j} \log z''_j. \tag{4.4.1}$$

Although Theorem 4.3.1 gives an isomorphism $\eta_1 \colon T_u U \to H^1(M; \mathscr{K})$ at totally incomplete hyperbolic structures, our construction of this map did not reveal the cohomological meaning of the derivative of the right-hand side of (4.4.1) with respect to u_l (or any other vector tangent to U at the point u) under η_1.

Geometrically, the calculation of e^{u_l} has been described by Thurston [50] and Neumann–Zagier [40] as a calculation of the *linear part* of the holonomy of the hyperbolic structure along u_l. In order to measure the linear part of the holonomy, we may consider the triangulated affine torus T_l determined by the shape parameters of the triangulation $\mathcal{T}$, treated here as variables. The holonomy of the affine structure of T_l along θ_l is a function on the gluing variety $\mathcal{V}^+_{\mathcal{T}}$ and can be written as a Möbius transformation

$$\omega \mapsto \alpha_l(z)\omega + \beta_l(z), \text{ where } \alpha_l(z) = e^{u_l(z)} \text{ and } z \in \mathcal{V}^+_{\mathcal{T}}. \tag{4.4.2}$$

To compute it in practice, one may use our groupoid description of holonomy of hyperbolic structures, as was in fact done in Example 2.3.9.

The completeness equation $u_l = 0$ implies that the linear part $\alpha_l(z)$ equals 1. In order to extract this linear part, Neumann–Zagier apply an analytic family of conjugating Möbius transformations, thus rewriting (4.4.2) as

$$\omega \mapsto \alpha_l(z)\omega$$

and ignoring β_l entirely (cf. Equation (31) in [40]). By studying the variation of α_l, Neumann and Zagier are essentially measuring how far the corresponding affine torus is from being a Euclidean torus, which is the case precisely when $u_l = 0$. Since we are only interested in a neighbourhood in $\mathcal{V}^+_{\mathcal{T}}$ of the point z_* corresponding to the unique complete hyperbolic structure on M, the torus T_l is Euclidean exactly when $\alpha_l(z) = 1$. Note that Neumann and Zagier consider two curves in each torus, which is not necessary if the tetrahedra are positively oriented, thanks to Remark 1.2.2.

In Section 1.3.1, we differentiated (4.4.1), obtaining equation (1.3.12). However, this procedure is not exactly equivalent to the study of an infinitesimal variation of the holonomy (4.4.2). Using Weil's method [54], we can associate to θ_l a cocycle

$$\frac{\partial}{\partial t} A_t A_0^{-1} \Big|_{t=0} \in \mathfrak{sl}_2\mathbb{C},$$

where $A_t \colon \omega \mapsto \alpha_l(y(u(t)))\omega + \beta_l(y(u(t)))$ and $t \mapsto u(t)$ is a 1-parameter family generated by

$v \in T_u U$. In particular, when $u = 0$, $A_0 = (\omega \mapsto \omega + \beta_l(z_*))$ is a pure translation, and we can express the product $A_t A_0^{-1}$ as

$$\big(\omega \mapsto \alpha_l(y(u))\omega + (\beta_l(y(u)) - \alpha_l(y(u))\beta_l(z_*))\big) \in \mathrm{Aff}(\mathbb{C}) \subset PSL_2\mathbb{C},$$

where $u = u(t)$ and $u(0) = 0$. The above affine map has the correct linear part, but it also contains an unwanted translational component. Craig Hodgson and Steven Kerckhoff explain on pp. 32–33 of [26] how to eliminate this translational component in a systematic way. Their method consists in projecting the cocycle to a certain subspace $\mathcal{A}$ and is therefore very geometric.

In our algebraic point of view, the projection which forgets the 'translational component' becomes just the usual quotient of cocycles by coboundaries. More precisely, we show that generically, at a totally incomplete hyperbolic structure, the unwanted translational part seen above corresponds to a coboundary, hence zero in cohomology. We then proceed to prove Theorem 4.4.3, which allows us to interpret derivatives of the log-parameters as cup products.

4.4.1 Understanding the map η_4

The map η_4 was defined in the proof of Theorem 4.3.1 by a diagram chase. At present, we wish to study the commutative square

$$\begin{array}{ccc} T_1\mathbb{C}^N_\times & \xrightarrow{Dp} & \mathbb{C}^k \\ \big\downarrow{\scriptstyle \eta_3} & & \big\downarrow{\scriptstyle \eta_4} \\ H^2(M;\mathscr{K}_{M,M_0}) & \xrightarrow{\pi} & H^2(M;\mathscr{K}) \end{array} \tag{4.4.3}$$

with the goal of understanding the precise algebraic and geometric meaning of the isomorphism $\eta_4\colon \mathbb{C}^k \to H^2(M;\mathscr{K})$.

In what follows, we shall restrict our attention to a totally incomplete hyperbolic structure. Since $H^2(M;\mathscr{K}) \cong H^2(\partial\bar{M};\mathscr{K}_{\partial\bar{M}})$, it suffices to study the tori $T_1,\dots,T_k$ at the ends of M. Fix an $l \in \{1,\dots,k\}$ and consider the torus T_l with its fundamental group given as $\langle a_1, a_2 \mid a_1a_2 = a_2a_1\rangle$. The holonomy representation ϱ can be conjugated so that

$$\varrho(a_1) = z \mapsto \alpha_1 z + \beta_1, \quad \varrho(a_2) = z \mapsto \alpha_2 z + \beta_2, \tag{4.4.4}$$

where $\alpha_1, \alpha_2 \neq 1$, as the hyperbolic structure is incomplete at each end. We may further assume $\beta_1, \beta_2 \neq 0$, because we are interested in a small deformation of the complete hyperbolic structure, where these conditions hold. Schematically, we can visualize the affine torus T_l, up to similarity,

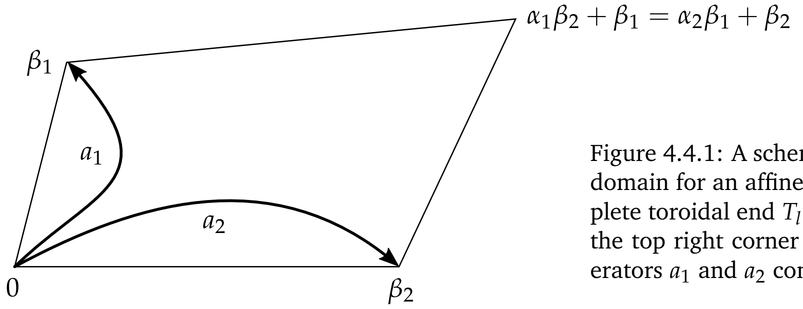


Figure 4.4.1: A schematic view of the fundamental domain for an affine torus associated to an incomplete toroidal end T_l of M. The condition stated at the top right corner must hold since the two generators a_1 and a_2 commute.

by taking a quadrilateral fundamental domain in $\mathbb{C}$ as shown in Figure 4.4.1.

Proposition 4.4.1. *Let L be an oriented geodesic spiralling towards an incomplete toroidal end T_l, $1 \leq l \leq k$. Equip T_l with a cellular structure comprising a single 2-cell. Then the cellular 2-cochain mapping the positively oriented 2-cell of T_l to the local Killing field $\mathfrak{h}_L$ of Proposition 1.1.5 defines a nontrivial cohomology class $[\mathfrak{h}]_l \in H^2(T_l; \mathscr{K})$. Moreover, this cohomology class does not depend on the choice of L.*

Proof. Consider the minimal CW-decomposition of the torus T_l with one 0-cell, two 1-cells corresponding to the generators a_1 and a_2 and oriented accordingly, and a single 2-cell R oriented using the convention 'outward facing normal vector in the last position'. We can assume that the 0-cell lies on the intersection of L with T_l. In order to compute $H^2(T_l; \mathscr{K})$, it suffices to consider $\mathfrak{sl}_2\mathbb{C}$-valued cochains with ϱ conjugated to the form (4.4.4). Using the standard basis $\{\mathfrak{e}, \mathfrak{h}, \mathfrak{f}\}$ of $\mathfrak{sl}_2\mathbb{C}$, we have a geometric basis of $C^1(T_l, \mathfrak{sl}_2\mathbb{C})$ given by the six cochains

$$a_1 \mapsto \mathfrak{e}, \quad a_1 \mapsto \mathfrak{h}, \quad a_1 \mapsto \mathfrak{f},$$
$$a_2 \mapsto \mathfrak{e}, \quad a_2 \mapsto \mathfrak{h}, \quad a_2 \mapsto \mathfrak{f},$$

whereas a geometric basis of $C^2(T_l, \mathfrak{sl}_2\mathbb{C})$ can similarly be formed from the three cochains

$$R \mapsto \mathfrak{e}, \quad R \mapsto \mathfrak{h}, \quad R \mapsto \mathfrak{f}.$$

Using formula (2.3.6), we see that $\operatorname{Ad} \varrho(a_i)$ has matrix representation

$$\operatorname{Ad} \varrho(a_i) \sim \begin{bmatrix} \alpha_i & -\beta_i & -\beta_i^2 \alpha_i^{-1} \\ 0 & 1 & 2\beta_i \alpha_i^{-1} \\ 0 & 0 & \alpha_i^{-1} \end{bmatrix}. \tag{4.4.5}$$

Hence, with respect to the bases chosen above, the coboundary map $\delta^1 \colon C^1(T_l; \mathfrak{sl}_2\mathbb{C}) \to C^2(T_l; \mathfrak{sl}_2\mathbb{C})$ has the matrix

$$\delta^1 \sim \begin{bmatrix} \alpha_1 - 1 & -\beta_1 & -\beta_1^2/\alpha_1 & \alpha_2 - 1 & -\beta_2 & -\beta_2^2/\alpha_2 \\ 0 & 0 & 2\beta_1/\alpha_1 & 0 & 0 & 2\beta_2/\alpha_2 \\ 0 & 0 & \alpha_1^{-1} - 1 & 0 & 0 & \alpha_2^{-1} - 1 \end{bmatrix}. \tag{4.4.6}$$

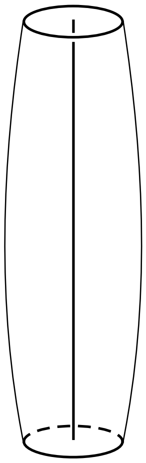
We are going to prove that the 2-cochain $R \mapsto \mathfrak{h}$ is not in the image of δ^1, which will show that it defines a non-trivial cohomology class $[\mathfrak{h}]_l \in H^2(T_l; \mathfrak{sl}_2\mathbb{C})$. Suppose to the contrary that there exist scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} -\beta_1^2/\alpha_1 \\ 2\beta_1/\alpha_1 \\ \alpha_1^{-1} - 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -\beta_2^2/\alpha_2 \\ 2\beta_2/\alpha_2 \\ \alpha_2^{-1} - 1 \end{pmatrix}. \tag{4.4.7}$$

Then in particular $\lambda_1(\alpha_1^{-1} - 1) + \lambda_2(\alpha_2^{-1} - 1) = 0$. As $\alpha_1\alpha_2 \neq 0$, this condition is equivalent to the equation

$$\lambda_1 \alpha_2 (1 - \alpha_1) + \lambda_2 \alpha_1 (1 - \alpha_2) = 0. \tag{4.4.8}$$

We now look at the second row of the right-hand side of (4.4.7) which we multiply by the non-



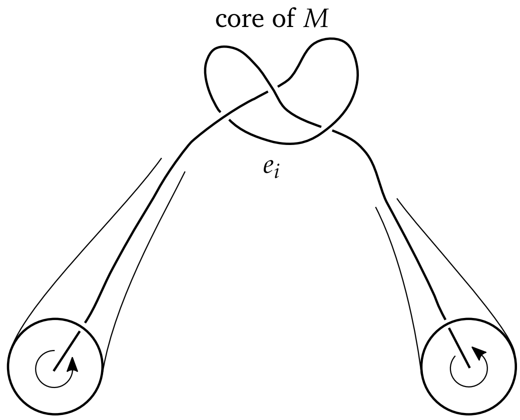


Figure 4.4.2: LEFT: An edge of an ideal triangulation meets the tori at the ends of M along two discs. RIGHT: When seen from the ends of M, the discs at the ends of the edge e_i appear positively oriented, since their orientations are induced by positive orientations on the tori. Nevertheless, no single orientation of e_i satisfies the right-hand rule on both discs.

zero factor $\alpha_1\alpha_2(1-\alpha_1)$, obtaining

$$2\alpha_1\alpha_2(1-\alpha_1)(\lambda_1\beta_1/\alpha_1+\lambda_2\beta_2/\alpha_2) = 2\big(\lambda_1\beta_1(1-\alpha_1)\alpha_2+\lambda_2\beta_2(1-\alpha_1)\alpha_1\big).$$

The commutativity relation of Figure 4.4.1 yields $\beta_2(1-\alpha_1) = \beta_1(1-\alpha_2)$. In this way, our expression becomes

$$2\beta_1\big(\lambda_1(1-\alpha_1)\alpha_2+\lambda_2(1-\alpha_2)\alpha_1\big),$$

which is zero by (4.4.8). Hence, the equation (4.4.7) is never satisfied.

It remains to prove the uniqueness of $[\mathfrak{h}]_l$. Suppose we act on the vector $\mathfrak{h} \in \mathfrak{sl}_2\mathbb{C}$ by the adjoint of an affine transformation $z \mapsto \alpha z + \beta$. From the matrix (4.4.5) we see that the image will be $\mathfrak{h} - \beta\mathfrak{e}$. On the other hand, since $\alpha_1, \alpha_2 \neq 1$, we observe from (4.4.6) that the basis vector $R \mapsto \mathfrak{e}$ is in the image of δ^1. Hence, the adjoint action of affine transformations changes the cochain $R \mapsto \mathfrak{h}$ at most by a coboundary. As a result, the cohomology class $[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K})$ is well defined and independent of the choice of the line L. Q.E.D.

Since the isomorphism $H^2(M;\mathscr{K}) \cong \prod_{l=1}^k H^2(T_l;\mathscr{K})$ is natural, we obtain the following characterization of the map η_4.

Proposition 4.4.2. *For every $l \in \{1,\dots,k\}$, the map η_4 sends the lth standard basis vector of $\mathbb{C}^k$ to the cohomology class $[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K})$ constructed in Proposition 4.4.1.*

Proof. Let $\eta_5\colon \mathbb{C}^k \to H^2(M;\mathscr{K})$ be the map taking, for all l, the lth standard basis vector of $\mathbb{C}^k$ to $[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K})$. In the proof of Theorem 4.3.1, we defined η_4 to be the unique map making the square (4.4.3) commutative, so in order to prove the equality $\eta_4 = \eta_5$, it suffices to show that $\eta_5 \circ Dp = \pi \circ \eta_3$. Recall from (1.3.7) that, with respect to the basis $\{\frac{\partial}{\partial x_i}\}_{i=1}^N \subset T_1\mathbb{C}_\times^N$, the map Dp has the matrix $K = [K_{li}]$ where each entry $K_{li} \in \{0,1,2\}$ records the number of ends of the edge e_i of the triangulation $\mathcal{T}$ incident to the lth end of M. Hence, we have

$$(\eta_5 \circ Dp)\left(\frac{\partial}{\partial x_i}\right) = \sum_{l=1}^k K_{li}[\mathfrak{h}]_l, \tag{4.4.9}$$

the sum regarded as an element of $H^2(M;\mathscr{K})$.

On the other hand, the basis element $\frac{\partial}{\partial x_i}$ is taken by η_3 to an element of $H^2(M;\mathscr{K}_{M,M_0})$ which by Proposition 4.1.3 can be identified with a Killing field acting as an infinitesimal translation along the edge e_i as soon as an orientation is specified.

The neighbourhood ν_i of the edge e_i intersects the tori $T_1, \dots, T_k$ along two discs (see Figure 4.4.2). As before, we orient these tori by the rule 'outward pointing normal vector in the last position', where outwards means towards the ends. With these orientations, the cohomology class $\eta_3(\frac{\partial}{\partial x_i})$ restricts to the germ of the unit-speed infinitesimal translation *towards infinity* on each of the two oriented discs (cf. Remark 4.1.4).

By Proposition 4.4.1, the local infinitesimal translation towards the end descends to the basis vector $[\mathfrak{h}]_l$ in the cohomology group $H^2(T_l; \mathscr{K})$. Hence, the image of $\pi(\eta_3(\frac{\partial}{\partial x_i}))$ in $H^2(M; \mathscr{K})$ is the sum of such contributions from both ends of e_i and can thus be written as $\sum_{l=1}^{k} K_{li}[\mathfrak{h}]_l$, which is exactly the right-hand side of (4.4.9). Q.E.D.

4.4.2 Log-parameters and cup products

Suppose that a multicurve θ satisfying Assumption 1.2.1 is fixed arbitrarily. We are going to take a fresh look at the derivatives of the corresponding log-parameters

$$\frac{\partial u_l}{\partial v} = \sum_{j=1}^{N} \left(G_{\theta_l,j} \frac{\partial \log z_j}{\partial z_j} + G'_{\theta_l,j} \frac{\partial \log z'_j}{\partial z_j} + G''_{\theta_l,j} \frac{\partial \log z''_j}{\partial z_j} \right) \frac{\partial z_j}{\partial v}, \tag{4.4.10}$$

where $v \in T_u U$ and $l \in \{1, \dots, k\}$; cf. (1.3.12). To improve notation, we introduce a complex linear map $L_\theta \colon T_z P^N \to \mathbb{C}^k$ defined on a basis element $\frac{\partial}{\partial z_j}$ by

$$L_\theta \Big(\frac{\partial}{\partial z_j} \Big) = \Big(G_{\theta_l,j} \zeta_j + G'_{\theta_l,j} \zeta'_j + G''_{\theta_l,j} \zeta''_j \Big)_{l=1}^{k}, \tag{4.4.11}$$

where we have used Notational Convention 1.3.3. With this notation, we have

$$\left(\frac{\partial u_l}{\partial v} \right)_{l=1}^{k} = (L_\theta \circ Dy)(v)$$

for any tangent vector $v \in T_u U$. In other words, the map L_θ encodes the simplicial computation of the log-parameters of the multicurve θ. The following theorem establishes a cohomological interpretation of this map.

Theorem 4.4.3. *Let θ and γ be two multicurves satisfying Assumption 1.2.1 and let $y = y_\theta$ be the map of* (1.3.5). *Suppose furthermore that $u \in U$ is a vector of small, non-zero log-parameters for the multicurve θ and consider M with the corresponding totally incomplete hyperbolic structure. Then we have a commutative diagram*

$$\begin{array}{ccccc} T_u U & \xrightarrow{\eta_1} & H^1(M; \mathscr{K}) & \xrightarrow{r^1} & H^1(M; \mathscr{K}_{\partial \bar{M}}) \\ {\scriptstyle L_\gamma \circ Dy} \big\downarrow & & & & \big\downarrow {\scriptstyle \smile [\gamma]^*} \\ \mathbb{C}^k & \xrightarrow{\eta_4} & H^2(M; \mathscr{K}) & \xrightarrow{r^2} & H^2(M; \mathscr{K}_{\partial \bar{M}}), \end{array}$$

where the map L_γ is defined by (4.4.11), *η_1 and η_4 are given by Theorem 4.3.1, the maps r^1 and r^2 are induced by restriction, and $\smile [\gamma]^* \colon H^1(M; \mathscr{K}_{\partial \bar{M}}) \to H^2(M; \mathscr{K}_{\partial \bar{M}})$ is the map given by the cup product with the Poincaré dual of the homology class of γ.*

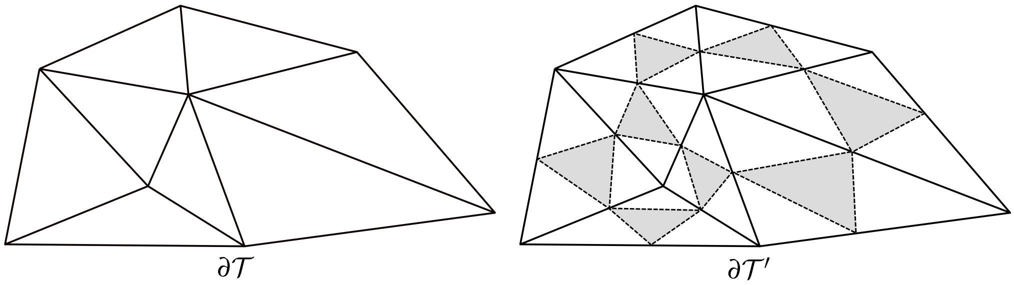


Figure 4.4.3: LEFT: A fragment of the triangulation $\partial\mathcal{T}$ of one of the tori at infinity of M induced by the geometric ideal triangulation $\mathcal{T}$. RIGHT: The triangulation $\partial\mathcal{T}'$ is obtained by subdividing each 2-simplex of $\partial\mathcal{T}$ into four 2-simplices. The central simplex of $\partial\mathcal{T}'$ in each triangle of $\partial\mathcal{T}$ is shaded. Normal segments are by definition the boundary simplices of the shaded triangles.

Note that in the above commutative diagram, the maps L_γ and $(\cdot \smile [\gamma]^*)$ depend on the homology class of γ, whereas η_1 and Dy depend on the homology class of θ.

Proof. Fix an index $l \in \{1, \ldots, k\}$ and denote by T_l the corresponding connected component of $\partial\bar{M}$. Let $\partial\mathcal{T}$ be the triangulation of the torus T_l into the truncation triangles of the ideal tetrahedra of $\mathcal{T}$. We further subdivide the triangulation $\partial\mathcal{T}$ into a triangulation $\partial\mathcal{T}'$ by inscribing an additional triangle into each truncation triangle, as shown in Figure 4.4.3. Following Futer and Guéritaud [19], we shall use the term *normal segments* to refer to the edges of $\partial\mathcal{T}'$ not contained in the edges of $\partial\mathcal{T}$.

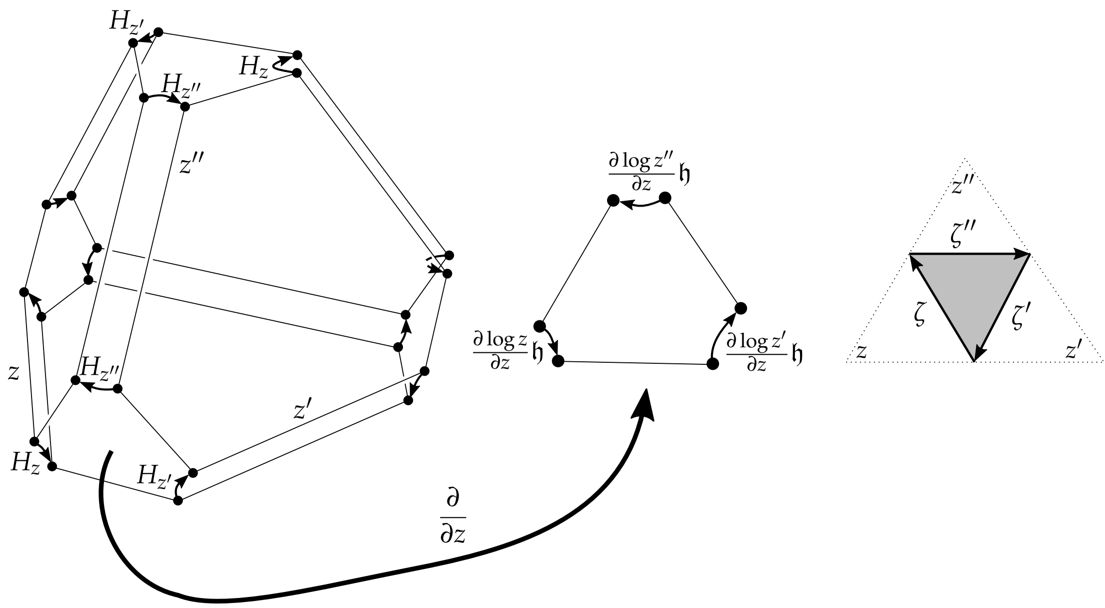
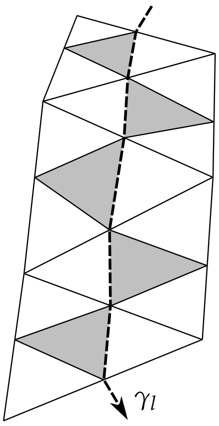


Figure 4.4.4: LEFT: The partial monodromies depending on the variable z are precisely those along the paths labeled H_z, $H_{z'}$, $H_{z''}$ within the tetrahedron Δ of shape parameter z. MIDDLE: Differentiating the monodromies with respect to the holomorphic parameter z, using Weil's method, results in deformation cocycles whose values on the directed arcs are multiples of the fields $\mathfrak{h}_e$ where e is the respective edge of the triangulation, oriented towards the end of M. RIGHT: By Proposition 4.4.1, each of these cocycles contributes a multiple of the cohomology class $[\mathfrak{h}]_l \in H^2(T_l; \mathscr{K})$, where T_l is the torus about the lth incomplete end of M. The coefficients in these multiples are precisely the numbers ζ, ζ', ζ'', as shown in the right panel. Note that the orientations of the simplices are opposite to the orientations of the groupoid paths, since we treat the holonomy as a *left* representation.

The homology class $[\gamma_l] \in H_1(T_l;\mathbb{C})$ can be represented simplicially as a sum of oriented normal segments in the truncation triangles which the curve $\gamma_l \subset T_l$ traverses, as depicted in Figure 4.4.5, left. This simplicial representative gives rise to the gluing equation coefficients $G^{\square}_{\gamma_l,j}$ which were introduced in Section 1.2.



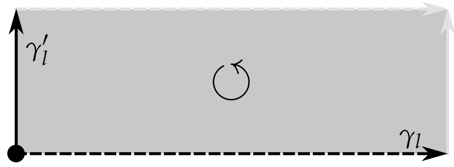


Figure 4.4.5: LEFT: The homology class of γ_l can be represented by a simplicial path in $\partial\mathcal{T}'$ consisting only of oriented normal segments. TOP RIGHT: Alternatively, the torus T_l can be modeled as a CW-complex in which γ_l corresponds to a 1-cell. The other 1-cell is then a curve γ_l' with the property $\iota([\gamma_l],[\gamma_l']) = 1$, where ι is the intersection pairing on $H_1(T_l;\mathbb{C})$.

Claim: We claim that the cup product $(r^1 \circ \eta_1)(\frac{\partial}{\partial u_l}) \smile [\gamma_l]^*$ is the image in the second cohomology of the 2-cochain whose value on the fundamental class of the torus T_l is the result of evaluating along γ_l a cocycle representing the cohomology class $(r^1 \circ \eta_1)(\frac{\partial}{\partial u_l})$. Indeed, consider for the moment the cell decomposition of the torus T_l with a single 0-cell, two 1-cells, and a single 2-cell. We can arrange so that γ_l is the closure of one of the oriented 1-cells; by an abuse of notation we call that cell γ_l. The other 1-cell can then be oriented in such a way that it represents a curve $\gamma_l' \subset T_l$ having intersection number 1 with γ_l. This CW-decomposition is depicted in the top right part of Figure 4.4.5. A cellular representative for the Poincaré dual $[\gamma_l]^*$ is then the cellular 1-cochain $\beta = (\gamma_l' \mapsto 1, \gamma_l \mapsto 0)$. Let α be a cellular 1-cocycle with coefficients in an arbitrary local system. Then the value of the cocycle $\alpha \smile \beta$ on the sole 2-cell reduces to the value of α on the cell γ_l, which finishes the proof of the claim. $\triangleleft$

Using the local computation presented in Figure 4.4.4, we obtain a simplicial 1-cocycle representing $(r^1 \circ \eta_1)(\frac{\partial}{\partial u_l})$, whose values on the oriented normal segments in the tetrahedron Δ_j are equal to $J_{jl}\zeta_j\mathfrak{h}$, $J_{jl}\zeta_j'\mathfrak{h}$, $J_{jl}\zeta_j''\mathfrak{h}$, where J is the Jacobian matrix studied in Section 1.3.1 and $\mathfrak{h}$ refers, in each case, to the unit speed infinitesimal translation along the respective edge of $\mathcal{T}$, towards infinity. Proposition 4.4.1 tells us that any such element $\mathfrak{h}$ becomes $[\mathfrak{h}]_l$ in $H^2(T_l;\mathscr{K})$. Hence, by counting the contributions from all triangles traversed by γ_l, we obtain

$$(r^1 \circ \eta_1)(\tfrac{\partial}{\partial u_l}) \smile [\gamma_l]^* = \sum_{j=1}^{N} (G_{\gamma_l,j}\zeta_j + G'_{\gamma_l,j}\zeta_j' + G''_{\gamma_l,j}\zeta_j'')J_{jl}[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K}). \tag{4.4.12}$$

Note that when $n \neq l$, we obtain trivially $(r^1 \circ \eta_1)(\frac{\partial}{\partial u_n}) \smile [\gamma_l]^* = 0$.

In order to understand $(r^2 \circ \eta_4 \circ L_\gamma \circ Dy)(\frac{\partial}{\partial u_l})$, it suffices to observe that $r^2 \circ \eta_4$ maps the lth standard basis vector of $\mathbb{C}^k$ to $[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K})$, which is the content of Proposition 4.4.2. Using (4.4.11) and the Jacobian matrix J of Dy, we immediately see that

$$(r^2 \circ \eta_4 \circ L_\gamma \circ Dy)(\tfrac{\partial}{\partial u_l}) = \sum_{j=1}^{N} (G_{\gamma_l,j}\zeta_j + G'_{\gamma_l,j}\zeta_j' + G''_{\gamma_l,j}\zeta_j'')J_{jl}[\mathfrak{h}]_l \in H^2(T_l;\mathscr{K}),$$

in agreement with (4.4.12). Hence, the diagram commutes. Q.E.D.

It turns out that the above theorem can be used to easily construct cohomology bases satisfying the balancing condition of Definition 3.5.1. This simplifies the computation of the non-abelian Reidemeister torsion.

Corollary 4.4.4. *Let M be an open 3-manifold admitting a complete hyperbolic structure of finite volume, but considered here with a deformed, totally incomplete structure with $k > 0$ incomplete ends and a corresponding small log-parameter u. If $\underline{b}$ is the standard basis of $\mathbb{C}^k$, then the basis*

$$\eta_1\left(\left\{\tfrac{\partial}{\partial u_1},\dots,\tfrac{\partial}{\partial u_k}\right\}\right)\cup\eta_4(\underline{b})\subset H^1(M;\mathscr{K})\times H^2(M;\mathscr{K})$$

is balanced with respect to the multicurve θ in the sense of Definition 3.5.1.

Proof. Write $\underline{b}=(b_1,\dots,b_k)$, with the ordering corresponding to the numbering of ends of M. We set $\gamma=\theta$ and wish to understand the matrix of the map $L_\theta\circ Dy_\theta$ in the bases $\left(\frac{\partial}{\partial u_l}\right)_{l=1}^k$ and $\underline{b}$. Since

$$L_\theta\circ D(y_\theta)\left(\frac{\partial}{\partial u_l}\right)=\left(\sum_{j=1}^N\left(G_{\theta_n,j}\frac{\partial\log z_j}{\partial z_j}+G'_{\theta_n,j}\frac{\partial\log z'_j}{\partial z_j}+G''_{\theta_n,j}\frac{\partial\log z''_j}{\partial z_j}\right)\frac{\partial z_j}{\partial u_l}\right)_{n=1}^k,$$

we see that the matrix of $L_\theta\circ Dy_\theta$ is

$$\left(G_\theta\operatorname{diag}(\zeta)+G'_\theta\operatorname{diag}(\zeta')+G''_\theta\operatorname{diag}(\zeta'')\right)\operatorname{Jac}y_\theta,$$

which is equal to the identity matrix by equation (1.3.15). Hence,

$$r^1\circ\eta_1\left(\tfrac{\partial}{\partial u_l}\right)\smile[\theta_n]^*=\delta_{ln}r^2\circ\eta_4(b_n),$$

where δ is the Kronecker symbol. In particular, we have the equality of sets

$$r^2(\eta_4(\underline{b}))=\bigcup_{l=1}^k\left\{r^1\left(\eta_1\left(\tfrac{\partial}{\partial u_l}\right)\right)\smile[\theta]^*\right\}.$$ Q.E.D.

As an application of Theorem 4.4.3, we explain how to recover Porti's change of curves formula [46, Proposition 4.7]. For simplicity, we focus on the case of a single toroidal end ($k=1$). Suppose that θ and $\widetilde{\theta}$ are two peripheral curves satisfying Assumption 1.2.1 and that $u,\widetilde{u}\neq 0$ are their log-parameters corresponding to the same deformed, incomplete hyperbolic structure on M. By Theorem 4.3.1, we have two maps $\eta_1\colon T_uU\xrightarrow{\cong}H^1(M;\mathscr{K})$ and $\widetilde{\eta}_1\colon T_{\widetilde{u}}\widetilde{U}\xrightarrow{\cong}H^1(M;\mathscr{K})$. Since both maps are isomorphisms of one-dimensional vector spaces, there exists a constant $c=c(\theta,\widetilde{\theta})$ such that $\widetilde{\eta}_1(\frac{\partial}{\partial\widetilde{u}})=c\eta_1(\frac{\partial}{\partial u})$. Hence, for every peripheral curve γ satisfying Assumption 1.2.1, we have

$$L_\gamma\circ Dy_{\widetilde{\theta}}\left(\frac{\partial}{\partial\widetilde{u}}\right)=cL_\gamma\circ Dy_\theta\left(\frac{\partial}{\partial u}\right).$$

In particular, for $\gamma=\theta$, (1.3.15) implies

$$c=cL_\theta\circ Dy_\theta\left(\frac{\partial}{\partial u}\right)=L_\theta\circ Dy_{\widetilde{\theta}}\left(\frac{\partial}{\partial\widetilde{u}}\right)=\sum_{j=1}^N\frac{\partial}{\partial\widetilde{u}}(G_{\theta,j}\zeta_j+G'_{\theta,j}\zeta'_j+G''_{\theta,j}\zeta''_j)=\frac{\partial u}{\partial\widetilde{u}}.$$

On the other hand, Corollary 4.4.4 says that the basis $\{\eta_1(\frac{\partial}{\partial u}),\eta_4(1)\}$ is balanced with respect to

θ, whereas $\{\widetilde{\eta}_1(\frac{\partial}{\partial \widetilde{u}}), \eta_4(1)\}$ is balanced with respect to $\widetilde{\theta}$. Using Theorem 3.1.4, we obtain

$$\frac{\mathbb{T}_{\mathrm{Ad}}(M,\theta)}{\mathbb{T}_{\mathrm{Ad}}(M,\widetilde{\theta})} = \det\left[\widetilde{\eta}_1\Big(\frac{\partial}{\partial \widetilde{u}}\Big) / \eta_1\Big(\frac{\partial}{\partial u}\Big)\right] = c.$$

Thus, we have

$$\mathbb{T}_{\mathrm{Ad}}(M,\theta) = \frac{\partial u}{\partial \widetilde{u}}\,\mathbb{T}_{\mathrm{Ad}}(M,\widetilde{\theta}). \tag{4.4.13}$$

Note that u and $\widetilde{u}$ can be viewed as multi-valued analytic functions on an open subset X_* of the character variety $X(M)$ containing the conjugacy class $[\varrho_*]$ of the discrete faithful representation. As usual, we arrange so that both u and $\widetilde{u}$ vanish at $[\varrho_*]$. Then u and $\widetilde{u}$ are both defined up to sign. Similarly, the torsion $\mathbb{T}_{\mathrm{Ad}}$ is an analytic function on X_*, also defined only up to sign [46]. Hence, although we only established (4.4.13) at totally incomplete hyperbolic structures, the equality extends to the complete structure by analytic continuation, recovering exactly the second part of Proposition 4.7 of [46].

We remark that when $(\theta, \widetilde{\theta}) = (\mu, \lambda)$ is a pair of generators of $H_1(\partial\bar{M};\mathbb{Z})$ with intersection number 1, the quantity $\frac{\partial u}{\partial \widetilde{u}}$, taken at the complete hyperbolic structure, can be interpreted geometrically [40] as the Euclidean shape of the torus at infinity of M, also known as the *cusp shape* of M.

Chapter 5

The 1-loop Conjecture of Dimofte and Garoufalidis

In this chapter, we wish to apply the results established so far to the study of the '1-loop Conjecture' of Tudor Dimofte and Stavros Garoufalidis [11]. In a nutshell, this conjecture postulates that the 'one-loop' coefficient of the proposed small $\hbar$ asymptotic expansion of the $SL_2\mathbb{C}$ Chern–Simons partition function $\mathcal{Z}_M(\hbar)$, defined combinatorially in terms of some geometric data describing a one-cusped hyperbolic 3-manifold M, is equal to the adjoint non-abelian Reidemeister torsion of M. We stress that it is not known whether the power series defined in [11] accurately describes the asymptotics of the Chern–Simons partition function. The rigorous definition of this partition function is complicated by major analytic difficulties stemming from the non-compactness of $SL_2\mathbb{C}$ and of M [1, 2]. Nevertheless, the '1-loop Conjecture' is unaffected by these issues and constitutes a valid mathematical question.

In [11], Dimofte and Garoufalidis focus primarily on the case when M is the complement of a hyperbolic knot in the 3-sphere. In this situation, the homology class of the knot meridian $[\mu] \in H_1(\partial M;\mathbb{Z})$ is well-defined up to sign, and the adjoint torsion can be calculated as $\mathbb{T}_{\mathrm{Ad}}(M,\mu)$.

The proposed asymptotic expansion of the partition function is a formal power series

$$S_0\hbar^{-1} - \frac{3}{2}\log\hbar + S_1 + S_2\hbar + S_3\hbar^2 + \cdots \qquad \text{(eq. 1–3 of [11])}$$

in which the coefficients $S_n \in \mathbb{C}$ are conjectured to be topological invariants of M, defined with some indeterminacy when $n = 1,2$ and unambiguously for $n > 2$. In particular, the definition of the above formal power series is engineered so that S_0 is the complex volume of M. As a consequence, the analogue of Kashaev's Volume Conjecture is satisfied automatically and the first non-trivial prediction concerning the above power series is that

$$\mathbb{T}_{\mathrm{Ad}}(M,\mu) \overset{?}{=} 4\pi^3 \exp(-2S_1). \qquad \text{(eq. 1–4 of [11])}$$

Fortunately, Dimofte and Garoufalidis propose a more explicit expression for the right-hand side of the above equality [11, Definition 1.2], which also makes sense both at the complete hyperbolic structure and at deformed, incomplete structures. We introduce this expression in Conjecture 5.1.5 below. Note that our statement of the 1-loop Conjecture is a generalization of the original statement in [11] in the sense that we allow M to be any hyperbolic 3-manifold with a single toroidal end (not necessarily a complement of a knot in S^3) and γ to be any homotopically nontrivial, simple, closed, peripheral curve in M.

After introducing the 1-loop Conjecture, we propose a strategy for a direct attack on the conjecture based on the material developed earlier in this work. Although our attack does not produce a complete proof, it succeeds in identifying the main term of the conjectural formula as a factor in an expression for the non-abelian torsion. In this way, we are able to reduce the 1-loop Conjecture to a simpler question concerning the Reidemeister torsion of a free group, which ought to give the remaining 'monomial correction' term of the conjectural formula. In particular, our approach confirms the nonvanishing of the conjectural formula and the correctness of its dependence on the choice of the peripheral curve γ.

We finish the chapter by providing additional information about the reduced conjecture and speculating on possible strategies which may lead to a complete answer to Dimofte–Garoufalidis' question. In particular, we propose a good parametrization of the geometries of the ideal tetrahedra which should further simplify the reduced conjecture.

5.1 Statement of the 1-loop Conjecture

Our goal in this section is to state a generalized version of the 1-loop Conjecture, which appeared in its original formulation as Conjecture 1.8 in [11].

5.1.1 Notations and definitions

Let $\mathcal{T}$ be a geometric ideal triangulation of M. As stated in Remark 1.3.2, in the case of a single cusp, any one of the edge consistency equations (1.2.3) is redundant – in other words, it is implied by the remaining $N-1$ equations. On the other hand, we have one completeness equation (1.2.7) written for a curve $\gamma = \theta_1$ satisfying Assumption 1.2.1. Therefore, following [11], we may remove the Nth edge equation and replace it by the completeness equation, thus keeping the number of equations at N. Of course, which equation is removed depends on the numbering of edges, but this fact does not affect the constructions that follow.

Define the matrices $\widehat{G}, \widehat{G}', \widehat{G}'' \in \mathcal{M}_{N\times N}(\mathbb{Z})$ by

$$\widehat{G}^{\square}_{ij} = \begin{cases} G^{\square}_{ij} & \text{if } 1 \le i \le N-1, \\ G^{\square}_{\gamma,j} & \text{if } i = N. \end{cases} \tag{5.1.1}$$

As discussed in Section 1.2, the solutions to (1.2.3) and (1.2.7) are also solutions of the system

$$\prod_{j=1}^{N} z_j^{\widehat{G}_{ij}} z_j'^{\widehat{G}'_{ij}} z_j''^{\widehat{G}''_{ij}} = 1, \quad i = 1, 2, \ldots, N. \tag{5.1.2}$$

In fact, both systems of equations are equivalent when restricted to a neighbourhood of the geometric solution $z_* \in \mathcal{V}^+_{\mathcal{T}} \subset P^N$.

Notational Convention 5.1.1. In what follows, we shall write $z = (z_1, \ldots, z_N)$ for the vector of shape parameters and extend this notation to analytic functions of z 'acting diagonally'.

More precisely, if $F\colon P \to \mathbb{C}$ is an analytic function, we write

$$F(z) \overset{\text{def}}{=\!=} \big(F(z_1), \dots, F(z_N)\big) \in \mathbb{C}^N.$$

For instance, the symbol $\frac{1}{z}$ will denote the vector $(\frac{1}{z_1}, \dots, \frac{1}{z_N})$. In particular, we shall write z' for $(z'_1, \dots, z'_N)$ and z'' for $(z''_1, \dots, z''_N)$.

Definition 5.1.2. A triple $f, f', f'' \in \mathbb{Z}^N$ of integer vectors satisfying

$$f + f' + f'' = (1, \dots, 1)^\top, \tag{5.1.3}$$

$$Gf + G'f' + G''f'' = (2, \dots, 2)^\top, \tag{5.1.4}$$

is called a *combinatorial flattening* on the triangulation $\mathcal{T}$ if for any homotopically non-trivial simple closed peripheral curve $\mu \subset \partial \bar{M}$ we have

$$G_\mu f + G'_\mu f' + G''_\mu f'' = 0. \tag{5.1.5}$$

Remark 5.1.3. The term *combinatorial flattening* was first used in [11], where the condition (5.1.5) was initially only imposed on a fixed meridian μ of a knot complement; a flattening satisfying the full condition (5.1.5) is nevertheless needed for the deformed version of the 1-loop Conjecture.

Remark 5.1.4. The existence of combinatorial flattenings satisfying a weaker peripheral condition had been proved by W. Neumann in [41] under relatively mild topological assumptions on the ideal triangulation $\mathcal{T}$ and then strengthened in [42, Theorem 4.5] to include condition (5.1.5) in its entirety. In particular, combinatorial flattenings always exist if $\mathcal{T}$ is a geometric ideal triangulation of a hyperbolic 3-manifold with rank two cusps. Note that the term 'flattening' is used in [42] in a somewhat different sense than here.

5.1.2 Statement of the conjecture

In the following, we assume that the 3-manifold M has only one toroidal end and is equipped with an ideal triangulation $\mathcal{T}$ with N tetrahedra. Moreover, we fix a curve γ satisfying Assumption 1.2.1.

Conjecture 5.1.5 (The 1-loop Conjecture)**.** Let $z \in \mathcal{V}^+_\mathcal{T}$ be a vector of shape parameters defining a small deformation of the complete hyperbolic structure on M. Then the non-abelian Reidemeister torsion of (M, γ) is given by the formula

$$\mathbb{T}_{\mathrm{Ad}}(M, \gamma) = \pm \frac{1}{2} \det\Big(\mathbf{A} \operatorname{diag}(z'') + \mathbf{B} \operatorname{diag}(z)^{-1}\Big) \times \prod_{j=1}^{N} z_j^{f''_j} z''^{-f_j}_j, \tag{5.1.6}$$

where (f, f', f'') is any combinatorial flattening, and the matrices $\mathbf{A}, \mathbf{B}$ are given by

$$\mathbf{A} = \widehat{G} - \widehat{G}', \qquad \mathbf{B} = \widehat{G}'' - \widehat{G}'. \tag{5.1.7}$$

Remark 5.1.6. The above conjecture can be viewed as postulating an equality of germs of regular functions on the $PSL_2\mathbb{C}$-character variety $X(M)$, although both functions are only defined up to sign. The germs are taken at the point $[\varrho_*]$, the conjugacy class of the holonomy representation of the complete structure.

Remark 5.1.7. The original statement of Conjecture 5.1.5 in [11] only postulates the equality (5.1.6) for $\gamma = \mu$, the meridian of a hyperbolic knot. Here, we are generalizing the conjecture to any peripheral curve γ satisfying Assumption 1.2.1 and do not require the one-cusped manifold M to be a knot complement. The dependence of the left-hand side of (5.1.6) on γ is via Porti's construction of a balanced basis (Definition 3.5.1), whereas the dependence of the right-hand side on γ is via the bottom rows of matrices $\widehat{G}, \widehat{G}', \widehat{G}''$ (see (5.1.1)).

Remark 5.1.8. It is unclear to the author what a flattening means geometrically. Recall that in (4.3.3), we defined a function

$$\ell \colon P^N \to \mathbb{C}^N, \quad \ell(z) = GZ + G'Z' + G''Z'' - (2\pi\sqrt{-1}, \ldots, 2\pi\sqrt{-1})^\top, \tag{5.1.8}$$

where we used standard branches of the logarithms Z, Z', Z''. The extra vector of $-2\pi\sqrt{-1}$'s was needed to shift the value $\ell(z_*)$ to zero. An alternative method of constructing ℓ is to instead shift the log-parameters themselves by defining

$$\widetilde{Z} = Z - \pi i f, \quad \widetilde{Z'} = Z' - \pi i f', \quad \widetilde{Z''} = Z'' - \pi i f'', \quad i = \sqrt{-1}.$$

Then we can put $\ell(z) = G\widetilde{Z} + G'\widetilde{Z'} + G''\widetilde{Z''}$, with $\ell(z_*) = 0$ now following directly from (1.2.6) and from the definition of a flattening.

Geometrically, the choice of the standard branches Z, Z', Z'' of the log-parameters is motivated by the fact that their imaginary parts are exactly the dihedral angles of the tetrahedra. However, it is more convenient algebraically to work with the homogenized logarithmic gluing equation $\ell(z) = 0$. A combinatorial flattening can therefore be viewed as an affine shift homogenizing the logarithmic gluing equations (1.2.6). This is reminiscent of the construction in [42], where a flattening is needed for a simplicial description of the classifying map $M \to BPSL_2\mathbb{C}$ of the étale bundle of the sheaf $\mathscr{D}$ (cf. Remark 2.2.4). Since the isomorphism class of this étale bundle is completely described by the homotopy class of its classifying map, it is plausible that combinatorial flattenings encode some interesting homological information about both.

Remark 5.1.9. We wish to stress that if the 1-loop Conjecture is true, it will provide the simplest known method of calculating the nonabelian Reidemeister torsion of a one-cusped hyperbolic 3-manifold. In fact, the right-hand side of (5.1.6) is an explicit expression depending only on the ordered quadruple $(\mathbf{A}, \mathbf{B}, z, (f, f', f''))$. This quadruple is called 'enhanced Neumann-Zagier datum' in [11]. The matrices $\mathbf{A}$ and $\mathbf{B}$ are obtained directly from the edge incidence numbers in the ideal triangulation $\mathcal{T}$ by the way of (5.1.7), whereas the shape parameters z and a combinatorial flattening (f, f', f'') are both solutions to certain kinds of 'gluing equations' on $\mathcal{T}$.

5.1.3 Symmetrization of the conjectural expression

We are now going to transform the right-hand side of (5.1.6) by reverting the substitutions (5.1.7). We use Notational Conventions 5.1.1 and 1.3.3.

We start by writing

$$\begin{aligned}
\mathbf{A}\,\mathrm{diag}\,(z'') + \mathbf{B}\,\mathrm{diag}\,(z)^{-1} &= (\widehat{G} - \widehat{G}')\,\mathrm{diag}\,(z'') + (\widehat{G}'' - \widehat{G}')\,\mathrm{diag}\,(z)^{-1} \\
&= \widehat{G}\,\mathrm{diag}\left(1 - \tfrac{1}{z}\right) - \widehat{G}' + \widehat{G}''\,\mathrm{diag}\left(\tfrac{1}{z}\right) \\
&= \left(\widehat{G}\,\mathrm{diag}\left(\tfrac{1}{z}\right) + \widehat{G}'\,\mathrm{diag}\left(\tfrac{1}{1-z}\right) + \widehat{G}''\,\mathrm{diag}\left(\tfrac{1}{z(z-1)}\right)\right)\mathrm{diag}\,(z-1) \\
&= \left(\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right)\mathrm{diag}\,(z-1)\,.
\end{aligned}$$

Taking determinants, we obtain

$$\begin{aligned}
\det\!\left(\mathbf{A}\,\mathrm{diag}\,(z'') + \mathbf{B}\,\mathrm{diag}\,(z)^{-1}\right) = \det\!\left(\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right) \\
\times \prod_{j=1}^{N}(z_j - 1). \qquad (5.1.9)
\end{aligned}$$

We group the term in the second line of (5.1.9) with the second factor of (5.1.6), which gives

$$\begin{aligned}
\prod_{j=1}^{N}(z_j - 1) z_j^{f_j''}\left(\frac{z_j - 1}{z_j}\right)^{-f_j} &= \prod_{j=1}^{N}(z_j - 1)^{1-f_j} z_j^{f_j''+f_j} \\
\text{– by (5.1.3) –}\quad &= \prod_{j=1}^{N}(z_j - 1)^{f_j''+f_j'} z_j^{f_j''+f_j} \\
&= \prod_{j=1}^{N}(-1)^{f_j'} z_j^{f_j}(1 - z_j)^{f_j'}\left(z_j(z_j - 1)\right)^{f_j''} \\
&= \pm z^{f}(1 - z)^{f'}\left(z(z-1)\right)^{f''} \\
&= \pm \zeta^{-f}\zeta'^{-f'}\zeta''^{-f''},
\end{aligned}$$

where the last two expressions use the multi-index notation. Collecting all terms that constitute the right-hand side of (5.1.6), we obtain an equivalent conjectural formula

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) \overset{?}{=} \pm \frac{\frac{1}{2}\det\!\left[\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right]}{\zeta^{f}\,\zeta'^{f'}\,\zeta''^{f''}}. \qquad (5.1.10)$$

We find the above form of Conjecture 5.1.5 more convenient to study. Furthermore, the formula on the right-hand side of (5.1.10) is arguably easier to remember.

5.2 Strategy of proof

In this section, we outline our plan of a direct attack on the 1-loop Conjecture 5.1.5. The notations are the same as in the preceding section.

The non-abelian Reidemeister torsion $\mathbb{T}_{\mathrm{Ad}}(M,\gamma)$ on the left-hand side of (5.1.10) can be computed in terms of a cellular complex whose geometric realisation in M is a retract of M. For this purpose, we shall use the 2-dimensional CW-complex described in Section 3.7.1, dual to the geometric ideal triangulation $\mathcal{T}$, which we assume to be fixed. Denoting this complex by X, we have

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = \mathbb{T}\left(C^\bullet(X;\mathscr{K}), \underline{c}_{\mathrm{geom}}, \underline{h(\gamma)}\right),$$

where $C^\bullet(X;\mathscr{K})$ is the cellular cochain complex with local coefficients in $\mathscr{K}$ and $\underline{h(\gamma)}$ is a basis of $H^\bullet(M;\mathscr{K})$ balanced with respect to γ. By the results of Chapter 3, we may take $\underline{c}_{\mathrm{geom}}$ to be a Steenrod geometric basis. In other words, $\underline{c}_{\mathrm{geom}}$ consists of cochains mapping oriented cells to pullbacks of fixed basis elements of $\mathscr{K}(\mathbb{H}^3) \cong \mathfrak{sl}_2\mathbb{C}$ under germs of orientation-preserving developing maps at the distinguished points of those cells. The fixed basis of $\mathfrak{sl}_2\mathbb{C}$ we shall be using is the basis $\{\mathfrak{e},\mathfrak{h},\mathfrak{f}\}$ defined in (1.1.4).

Note that the closed subspace M_0 studied in Section 4.1 retracts onto the 1-skeleton $X^{(1)} \subset X$. Therefore, the short exact sequence of sheaves

$$0 \to \mathscr{K}_{M,M_0} \to \mathscr{K} \to \mathscr{K}_{M_0} \to 0 \tag{5.2.1}$$

can be understood in terms of the short exact sequence of cellular cochain complexes associated to the pair (M, M_0), or, in our case, $(X, X^{(1)})$. This short exact sequence can be written as

$$0 \to C^\bullet(X, X^{(1)};\mathscr{K}) \to C^\bullet(X;\mathscr{K}) \to C^\bullet(X^{(1)};\mathscr{K}) \to 0. \tag{5.2.2}$$

Applying Theorem 3.1.5 to (5.2.2), we obtain

$$\begin{aligned}\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = {} & \mathbb{T}\left(C^\bullet(X,X^{(1)};\mathscr{K}), \underline{c}^{(2)}_{\mathrm{geom}}, \underline{h}_{\mathrm{rel}}\right)\mathbb{T}\left(C^\bullet(X^{(1)};\mathscr{K}), \underline{c}^{(0,1)}_{\mathrm{geom}}, \underline{h}_{M_0}\right) \\ & \times \mathbb{T}\left(\mathcal{H}, \underline{h(\gamma)} \cup \underline{h}_{\mathrm{rel}} \cup \underline{h}_{M_0}, \varnothing\right),\end{aligned} \tag{5.2.3}$$

where $\mathcal{H}$ is the long exact sequence in cohomology associated to (5.2.1), $\underline{c}^{(2)}_{\mathrm{geom}}$ is induced by a Steenrod geometric basis on the 2-cells of X, and $\underline{c}^{(0,1)}_{\mathrm{geom}}$ is a Steenrod geometric basis on the 1-skeleton $X^{(1)}$. Note that these geometric bases satisfy Milnor's compatibility condition automatically. In the above formula, $\underline{h}_{\mathrm{rel}}$ can be any basis of $H^\bullet(M;\mathscr{K}_{M,M_0})$, whereas $\underline{h}_{M_0}$ can be any basis of $H^\bullet(M;\mathscr{K}_{M_0})$.

The long exact sequence $\mathcal{H}$ is none other than (4.1.4), which we write as

$$\mathcal{H}^\bullet: \qquad 0 \to \underset{4}{H^1(M;\mathscr{K})} \to \underset{5}{H^1(M;\mathscr{K}_{M_0})} \to \underset{6}{H^2(M;\mathscr{K}_{M,M_0})} \to \underset{7}{H^2(M;\mathscr{K})} \to 0, \tag{5.2.4}$$

with the grading given by (3.1.4) indicated under each term. It is with this grading (up to even shifts) that we need to compute the last term of (5.2.3).

Under the assumption that the hyperbolic structure on M is totally incomplete, Theorem 4.3.1 guarantees that the acyclic cochain complex $\mathcal{H}^\bullet$ fits into the following short exact sequence of acyclic cochain complexes:

$$0 \to T\mathcal{GE} \xrightarrow{\eta} \mathcal{H} \to \operatorname{Coker}\eta \to 0. \tag{5.2.5}$$

We wish to apply Milnor's multiplicativity theorem (Theorem 3.1.5) again. Therefore, we need to construct bases of the cochain complexes $T\mathcal{GE}$ and $\operatorname{Coker}\eta$ compatible with the basis $\underline{h(\gamma)} \cup \underline{h}_{\mathrm{rel}} \cup \underline{h}_{M_0}$. Note that the bases $\underline{h}_{\mathrm{rel}}$ and $\underline{h}_{M_0}$ have not been specified yet and can be chosen freely. On the other hand, the basis $\underline{h(\gamma)}$ cannot be chosen freely, as it needs to be balanced with respect to γ in the sense of Definition 3.5.1. Since we work with a totally incomplete hyperbolic structure, we may use Corollary 4.4.4, concluding that the basis $\{\eta_1(\frac{\partial}{\partial u}), \eta_4(1)\}$ is balanced with respect to γ, where $u \neq 0$ is the log-parameter of γ. Therefore, we can use the bases of $T\mathcal{GE}$ listed in Section 1.3.1. We set

$$\underline{h}_{M_0} = \eta_2(\{\tfrac{\partial}{\partial z_1}, \dots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{h'}_{M_0} \quad \text{and} \quad \underline{h}_{\mathrm{rel}} = \eta_3(\{\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_N}\}) \cup \underline{h'}_{\mathrm{rel}}, \tag{5.2.6}$$

where $\underline{h'}_{M_0}$ and $\underline{h'}_{\mathrm{rel}}$ have yet to be defined; they need to be collections of linearly independent vectors spanning a subspace complementary to the image of η in $\mathcal{H}$. By an abuse of notation, we use the same symbols to denote the images of $\underline{h'}_{M_0}$ and $\underline{h'}_{\mathrm{rel}}$ in Coker η_2 and Coker η_3, respectively. In this way, we have

$$\begin{aligned}\mathbb{T}\left(\mathcal{H}, \underline{h(\gamma)} \cup \underline{h}_{\mathrm{rel}} \cup \underline{h}_{M_0}, \varnothing\right) = \mathbb{T}\Big(T\mathcal{GE}, \left\{\tfrac{\partial}{\partial u}\right\} \cup \left\{\tfrac{\partial}{\partial z_j}\right\}_j \cup \left\{\tfrac{\partial}{\partial x_i}\right\}_i \cup \{1\}, \varnothing\Big) \\ \times \mathbb{T}\left(\operatorname{Coker}\eta, \underline{h'}_{M_0} \cup \underline{h'}_{\mathrm{rel}}, \varnothing\right).\end{aligned}$$

Substituting this into (5.2.3), we obtain

$$\begin{aligned}\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = \mathbb{T}\left(C^\bullet(X, X^{(1)}; \mathscr{K}), \underline{c}^{(2)}_{\mathrm{geom}}, \underline{h}_{\mathrm{rel}}\right) \mathbb{T}\left(C^\bullet(X^{(1)}; \mathscr{K}), \underline{c}^{(0,1)}_{\mathrm{geom}}, \underline{h}_{M_0}\right) \\ \times \mathbb{T}\Big(T\mathcal{GE}, \left\{\tfrac{\partial}{\partial u}\right\} \cup \left\{\tfrac{\partial}{\partial z_j}\right\}_j \cup \left\{\tfrac{\partial}{\partial x_i}\right\}_i \cup \{1\}, \varnothing\Big) \mathbb{T}\left(\operatorname{Coker}\eta, \underline{h'}_{M_0} \cup \underline{h'}_{\mathrm{rel}}, \varnothing\right).\end{aligned} \tag{5.2.7}$$

Note that although the above equality holds only at totally incomplete structures, we can use analytic continuation to extend it to all finite volume hyperbolic structures, including the unique complete structure.

In the following sections, we are going to analyse the four terms making up the right-hand side of (5.2.7). As we are going to see, only two of them contribute nontrivially to the answer.

5.3 Torsion of the relative subcomplex

At present, we are interested in calculating the torsion $\mathbb{T}\left(C^\bullet(X, X^{(1)}; \mathscr{K}), \underline{c}^{(2)}_{\mathrm{geom}}, \underline{h}_{\mathrm{rel}}\right)$, which constitutes the first term of the decomposition (5.2.7). We continue using the notations of the previous section.

The relative complex $C^\bullet(X, X^{(1)}; \mathscr{K})$ computes the cohomology of the subsheaf $\mathscr{K}_{M,M_0} \subset \mathscr{K}$. We recall that in Section 4.1.2 we established the isomorphism $H^2(M; \mathscr{K}_{M,M_0}) \cong \prod_{i=1}^N \mathscr{K}(\nu_i)$, where $\nu_i \subset M \setminus M_0$ is an open neighbourhood of the ith edge of the triangulation $\mathcal{T}$. By Proposition 4.1.2, all remaining cohomology groups of $\mathscr{K}_{M,M_0}$ vanish. Since $X^{(1)}$ is the 1-skeleton of the 2-complex X, the cochain complex $C^\bullet(X, X^{(1)}; \mathscr{K})$ reduces to a single nonzero term,

$$0 \to C^2(X, X^{(1)}; \mathscr{K}) \to 0,$$

which is therefore canonically isomorphic to $H^2(M; \mathscr{K}_{M,M_0})$. Suppose s_i is an oriented 2-cell of X dual to the edge e_i (see Figure 3.7.1). There is a unique orientation of the edge e_i which satisfies the right-hand rule with respect to the chosen orientation of s_i (cf. Remark 4.1.4). Let $\varphi_i \colon \nu_i \to \mathbb{H}^3$ be any geometric coordinate chart taking e_i with this orientation to the oriented geodesic $\overrightarrow{(0,\infty)} \subset \mathbb{H}^3$. Consider the cellular 2-cochains defined by

$$\alpha(i,v)\colon \begin{cases} s_i \mapsto (\varphi_i)_*^{-1} v, \\ s_j \mapsto 0 \text{ if } i \neq j, \end{cases} \quad \text{where } v \in \mathfrak{sl}_2\mathbb{C} = \mathscr{K}(\mathbb{H}^3). \tag{5.3.1}$$

Then the set

$$\underline{c}^{(2)}_{\text{geom}} \overset{\text{def}}{=\!=} \bigcup_{i=1}^{N} \{\alpha(i,\mathfrak{e}), \alpha(i,\mathfrak{h}), \alpha(i,\mathfrak{f})\} \subset C^2(X, X^{(1)}; \mathscr{K})$$

is a geometric basis in the sense of Steenrod. Moreover, thanks to the canonical isomorphism $H^2(M; \mathscr{K}_{M,M_0}) \cong C^2(X, X^{(1)}; \mathscr{K})$, we can view $\underline{c}^{(2)}_{\text{geom}}$ as a basis of $H^2(M; \mathscr{K}_{M,M_0})$. Observe that, by definition of the map η_3 in the proof of Theorem 4.3.1, we have $\eta_3(\frac{\partial}{\partial x_i}) = \pm\alpha(i,\mathfrak{h})$ for every $i \in \{1, \ldots, N\}$. Therefore, we can set

$$\underline{h}'_{\text{rel}} \overset{\text{def}}{=\!=} \bigcup_{i=1}^{N} \{\alpha(i,\mathfrak{e}), \alpha(i,\mathfrak{f})\} \quad \text{and} \quad \underline{h}_{\text{rel}} \overset{\text{def}}{=\!=} \eta_3(\{\tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_N}\}) \cup \underline{h}'_{\text{rel}}. \tag{5.3.2}$$

In this way, the basis $\underline{h}_{\text{rel}}$ is compatible with the bases $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\} \subset T_1\mathbb{C}^N_\times$ and the image of $\underline{h}'_{\text{rel}}$ in $\operatorname{Coker} \eta_3$. At the same time, we have

$$\mathbb{T}\left(C^\bullet(X, X^{(1)}; \mathscr{K}), \underline{c}^{(2)}_{\text{geom}}, \underline{h}_{\text{rel}}\right) = \pm\det\left[\underline{h}_{\text{rel}} / \underline{c}^{(2)}_{\text{geom}}\right] = \pm 1. \tag{5.3.3}$$

Note that we have defined $\underline{h}'_{\text{rel}}$ in (5.3.2) and now must use the image of this $\underline{h}'_{\text{rel}}$ in $\operatorname{Coker} \eta_3$ when computing the last term of (5.2.7).

Remark 5.3.1. The basis $\underline{h}'_{\text{rel}}$ defined in (5.3.2) is not unique, since it depends on the following choices: 1) the choice of an orientation of each 2-cell s_i; 2) the choice of the geometric chart φ_i for every i. However, changing the orientations of cells only affects the sign of the corresponding local Killing vector fields, which has no effect on Reidemeister torsion due to its inherent sign indeterminacy. Moreover, as seen from (1.1.9), changing the geometric chart φ_i results in replacing $\alpha(i,\mathfrak{e})$ with $\lambda\alpha(i,\mathfrak{e})$ and $\alpha(i,\mathfrak{f})$ with $\lambda^{-1}\alpha(i,\mathfrak{f})$ for the same constant $\lambda \in \mathbb{C}_\times$. Hence, the exterior product $\alpha(i,\mathfrak{e}) \wedge \alpha(i,\mathfrak{f}) \in \bigwedge^2 H^2(M; \mathscr{K}_{M,M_0})$ is in fact well-defined up to sign. As a consequence, the basis $\underline{h}'_{\text{rel}}$ always maps to the same element of the top exterior power of $\operatorname{Coker} \eta_3$ (up to sign), irrespectively of the choices mentioned above.

5.4 Torsion of the cokernel complex

Our goal is now to compute the torsion $\mathbb{T}\left(\operatorname{Coker} \eta, \underline{h}'_{M_0} \cup \underline{h}'_{\text{rel}}, \varnothing\right)$ which constitutes the last term of (5.2.7). We continue using the notations of Section 5.2.

The cokernel complex $\operatorname{Coker} \eta$ of Theorem 4.3.1 consists of only two nonzero terms

$$0 \to \underset{5}{\operatorname{Coker} \eta_2} \xrightarrow{[\partial]} \underset{6}{\operatorname{Coker} \eta_3} \to 0,$$

with the only nontrivial map being the isomorphism $[\partial]$ induced on the cokernels by the connecting homomorphism $\partial\colon H^1(M; \mathscr{K}_{M_0}) \to H^2(M, \mathscr{K}_{M,M_0})$. The prescribed basis of $\operatorname{Coker} \eta_3$ is the image of $\underline{h}'_{\text{rel}}$ as defined in (5.3.2).

Since $[\partial]$ is an isomorphism, we may define

$$\underline{h'}_{M_0} \stackrel{\text{def}}{=\!=} [\partial]^{-1}(\underline{h'}_{\mathrm{rel}}) \subset \operatorname{Coker}\eta_2. \tag{5.4.1}$$

This choice is designed to ensure that we have

$$\mathbb{T}\left(\operatorname{Coker}\eta, \underline{h'}_{M_0} \cup \underline{h'}_{\mathrm{rel}}, \varnothing\right) = \det\left[[\partial]^{-1}(\underline{h'}_{\mathrm{rel}})/\underline{h'}_{M_0}\right]^{-1} = \pm 1. \tag{5.4.2}$$

To ensure the compatibility of bases in (5.2.5), the basis $\underline{h'}_{M_0}$ chosen in (5.4.1) now needs to be substituted into (5.2.6). More precisely, we need to find a preimage of $\underline{h'}_{M_0} \subset \operatorname{Coker}\eta_2$ under the quotient map $H^1(M; \mathscr{K}_{M_0}) \to \operatorname{Coker}\eta_2$.

5.4.1 Lifting the basis of the cokernel

We turn to the study of the problem of lifting the basis $\underline{h'}_{M_0}$ defined in (5.4.1) to $H^1(M; \mathscr{K}_{M_0})$. Recall from the proof of Theorem 4.3.1 that the connecting homomorphism $\partial\colon H^1(M; \mathscr{K}_{M_0}) \to H^2(M, \mathscr{K}_{M,M_0})$ is essentially the evaluation of a cohomology class on the boundary cycles in M_0 dual to the removed edges of $\mathcal{T}$.

To obtain a geometric interpretation of $\underline{h'}_{M_0}$, we orient every edge e_i of $\mathcal{T}$ arbitrarily and consider the compatibly oriented cycle $\varepsilon_i \in H_1(M_0; \mathbb{Z})$ depicted on the left panel of Figure 4.3.1. In the proof of Theorem 4.3.1, we only evaluated on ε_i the cohomology classes with values in the subspace span $\mathfrak{h}_{e_i} \subset \mathscr{K}(v_i)$, so the result did not depend on the choice of a local coordinate chart sending e_i to the oriented geodesic $\overrightarrow{(0,\infty)} \subset \mathbb{H}^3$.

In order to lift the basis $\underline{h'}_{M_0} \subset \operatorname{Coker}\eta_2$ to the cohomology group $H^1(M; \mathscr{K}_{M_0})$, it suffices to find a linearly independent set

$$\underline{\beta} = \bigcup_{i=1}^{N} \{\beta(i,\mathfrak{e}), \beta(i,\mathfrak{f})\} \subset H^1(M; \mathscr{K}_{M_0}) \tag{5.4.3}$$

whose elements satisfy

$$\partial(\beta(i,\mathfrak{e})) = \alpha(i,\mathfrak{e}) \quad \text{and} \quad \partial(\beta(i,\mathfrak{f})) = \alpha(i,\mathfrak{f}) \qquad \text{for all } i \in \{1,\ldots,N\}, \tag{5.4.4}$$

where the α's above are those of (5.3.1).

As explained in Remark 5.3.1, the cohomology classes $\alpha(i,\mathfrak{e})$, $\alpha(i,\mathfrak{f})$ are not defined uniquely, even though their exterior product

$$\alpha(i,\mathfrak{e}) \wedge \alpha(i,\mathfrak{f}) \in \bigwedge^2 H^2(M; \mathscr{K}_{M,M_0}) \cong \bigwedge^2 \mathscr{K}(v_i)$$

is well defined up to sign. Hence, the preimage classes $\beta(i,\mathfrak{e}), \beta(i,\mathfrak{f}) \in H^1(M; \mathscr{K}_{M_0})$ will also fail to be unique.

Since the Killing vector fields $\mathfrak{e}$ and $\mathfrak{f}$ act as infinitesimal translations in horospheres at the two ends of e_i (cf. Figure 1.1.1), the classes $\beta(i,\mathfrak{e}), \beta(i,\mathfrak{f})$ can be interpreted geometrically as infinitesimal shearing deformations of the cycle ε_i. One such deformation is shown pictorially in Figure 5.4.1.

Suppose we fix a basepoint $a_i \in \operatorname{Ob}(\mathcal{G}(\mathcal{T}))$ and represent the homotopy class of ε_i, based at a_i, as a product of morphisms of $\mathcal{G}(\mathcal{T})$. Then the partial monodromy of a hyperbolic structure

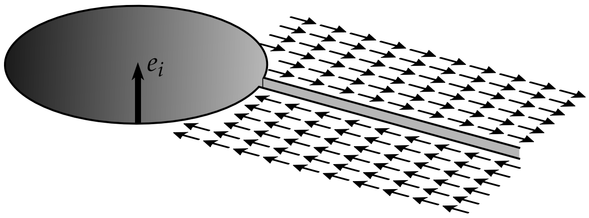


Figure 5.4.1: An “infinitesimal earthquake” $\beta(i,\mathfrak{e}) \in H^1(M;\mathscr{K}_{M_0})$ is visualized near the end of a removed edge e_i of the triangulation $\mathcal{T}$. Since the pairings of the faces of $\mathcal{T}$ are geometrically rigid, the earthquake is not a tangent vector to any shape deformation of $\mathcal{T}$, but rather rips $\mathcal{T}$ apart along the fault line.

on M_0 determined by a collection of shape parameters $z \in P^N$ along ε_i equals $H_{g_i(z)}$, where g_i is the ith component of the gluing map defined in (1.2.8). Obviously, for every $s \in \mathbb{C}_\times$ the Möbius transformation H_s given in (2.3.2) preserves the axis $(0,\infty) \in \mathbb{H}^3$. We can view the presence of this invariant axis as a property of every flat $PSL_2\mathbb{C}$ bundle on M_0 associated to a point $z \in P^N$. This property is in a sense a by-product of Thurston's construction of flat bundles via gluings of hyperbolic ideal tetrahedra, which we discussed in Section 4.2. Viewed purely algebraically, a flat bundle on M_0 is the same as a conjugacy class of a representation of the free group $F_{N+1} \cong \pi_1(M_0)$ in $PSL_2\mathbb{C}$. Since the cohomology group $H^1(M;\mathscr{K}_{M_0})$ is isomorphic to the tangent space to the whole $PSL_2\mathbb{C}$ character variety of M_0, the exterior product

$$\operatorname{vol}\underline{\beta} \stackrel{\text{def}}{=\!=} \bigwedge_{i=1}^{N} \beta(i,\mathfrak{e}) \wedge \beta(i,\mathfrak{f})$$

can be viewed as an unsigned ‘volume’ on a subspace of $H^1(M;\mathscr{K}_{M_0})$ complementary to $\operatorname{Im}\eta_2$, hence complementary to those infinitesimal deformations of flat bundles on M_0 which stem from the variation of the shape parameters of $\mathcal{T}$. In other words, the cohomology classes $\beta(i,\mathfrak{e}), \beta(i,\mathfrak{f})$ can be seen as unit deformations of the conjugacy classes of $PSL_2\mathbb{C}$ representations *away from* the image of the map E introduced in (4.2.4).

5.5 Torsion of the gluing complex

In this section, we calculate the torsion of the ‘gluing complex’

$$T\mathcal{GE}: \qquad 0 \to T_uU \xrightarrow{Dy} T_zP^N \xrightarrow{Dg} T_1\mathbb{C}_\times^N \xrightarrow{Dp} \mathbb{C}^k \to 0 \qquad (5.5.1)$$

which was studied in Section 1.3, with respect to the bases defined on p. 38. At first, we carry out the calculation in the case of a single toroidal end. This case has the most relevance to Conjecture 5.1.5. We then proceed to indicate how one may generalize this calculation to the case of any $k > 1$, although the argument leading to an explicit formula in the general case requires an additional assumption on the ideal triangulation.

5.5.1 The case of a single end

At first, we assume that M has a single end, i.e., $k = 1$. We denote by γ an arbitrarily fixed curve satisfying Assumption 1.2.1. Thus, γ can be used to write down the completeness equation (1.2.7) on M, thereby defining the map $y = y_\gamma$. We treat the ‘gluing complex’ (1.3.6) as an acyclic cochain complex

$$T\mathcal{GE}\Big|_{k=1}: \qquad \underset{4}{0 \to T_u U} \xrightarrow{Dy} \underset{5}{T_z P^N} \xrightarrow{Dg} \underset{6}{T_1 \mathbb{C}_\times^N} \xrightarrow{Dp} \underset{7}{\mathbb{C}} \to 0, \tag{5.5.2}$$

with the grading indicated under each term. Note that this is precisely the grading induced via (5.2.5) from (5.2.4).

Recall that the matrix expressions for the maps ocurring in (5.5.2) were computed in Section 1.3.1. These results can be summarized as follows:

$$\begin{aligned} \operatorname{Jac} p &= [2\ \ 2\ \cdots\ 2]\,, \\ \operatorname{Jac} g &= G\operatorname{diag}(\zeta) + G'\operatorname{diag}(\zeta') + G''\operatorname{diag}(\zeta'')\,, \\ 1 &= \Big(G_\gamma \operatorname{diag}(\zeta) + G'_\gamma \operatorname{diag}(\zeta') + G''_\gamma \operatorname{diag}(\zeta'')\Big)\operatorname{Jac} y. \end{aligned} \tag{5.5.3}$$

Indeed, the first equality results from combining Remark 1.3.1 with (1.3.7), the second is simply (1.3.10), whereas the third equality is (1.3.15) for $k = 1$, where G_γ, G'_γ, G''_γ are the row vectors of integers introduced in (1.3.11).

We start the calculation of torsion from the right, choosing for the final term $\mathbb{C}$ the basis $\underline{b^7} = \underline{c^7} = \{1\}$, so that the change-of-basis matrix equals $A_7 = [1]$.

Next, we choose the pre-image of the vector $1 \in \mathbb{C}$ under Dp to be $\frac{1}{2}\frac{\partial}{\partial x_N} \in T_1\mathbb{C}_\times^N$. A completion of this vector to a basis of $T_1\mathbb{C}_\times^N$ is given by the collection of vectors $\left\{\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_N}\right\}_{i=1,\ldots,N-1}$. Expressing this basis in terms of the original basis $\{\frac{\partial}{\partial x_i}\}_i$, we obtain the change-of-basis matrix

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ -1 & -1 & -1 & \ldots & -1 & \frac{1}{2} \end{bmatrix}. \tag{5.5.4}$$

Note that each vector $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_N}$ is an element of $\operatorname{Ker} Dp = \operatorname{Im} Dg$. Therefore, it is possible to choose pre-images of these vectors in $T_z P^N$,

$$v_i \in (Dg)^{-1}\left(\left\{\tfrac{\partial}{\partial x_i} - \tfrac{\partial}{\partial x_N}\right\}\right), \quad 1 \le i \le N-1. \tag{5.5.5}$$

We complete $\{v_1, \ldots, v_{N-1}\}$ to a basis of $T_z P^N$ by adjoining the vector

$$Y := Dy\left(\frac{\partial}{\partial u}\right) = \sum_{j=1}^{N} [\operatorname{Jac} y]_{j,1}\,\frac{\partial}{\partial z_j}.$$

By the very definition of Y, the coefficients of the expression of Y in the basis $\left\{\frac{\partial}{\partial z_j}\right\}_j$ of $T_z P^N$ are given by the sole column vector of $\operatorname{Jac} y$. Moreover, $Dg(Y) = Dg\left(Dy\left(\frac{\partial}{\partial u}\right)\right) = 0$. In this way, the change-of-basis matrix at $T_z P^N$ becomes

$$A_5 = \begin{bmatrix} | & | & & | & \boxed{} \\ v_1 & v_2 & \cdots & v_{N-1} & \operatorname{Jac} y \\ | & | & & | & \boxed{} \end{bmatrix},$$

with each column vector marked v_i containing the coefficients of v_i in the basis $\{\frac{\partial}{\partial z_j}\}_j$. Let $\widehat{G}$, $\widehat{G}'$, $\widehat{G}''$ be as in (5.1.1); recall that in our case these matrices differ from the corresponding gluing matrices G, G' and G'' only in their last rows. In paricular, we have

$$\widehat{\mathbb{G}} := \widehat{G} \operatorname{diag}(\zeta) + \widehat{G}' \operatorname{diag}(\zeta') + \widehat{G}'' \operatorname{diag}(\zeta'') = \begin{bmatrix} \boxed{\begin{matrix} \text{rows } 1, \ldots, N-1 \text{ of} \\ G \operatorname{diag}(\zeta) + G' \operatorname{diag}(\zeta') + G'' \operatorname{diag}(\zeta'') \end{matrix}} \\ G_\gamma \operatorname{diag}(\zeta) + G'_\gamma \operatorname{diag}(\zeta') + G''_\gamma \operatorname{diag}(\zeta'') \end{bmatrix}.$$

Since $G \operatorname{diag}(\zeta) + G' \operatorname{diag}(\zeta') + G'' \operatorname{diag}(\zeta'')$ is the matrix of Dg, we see from (5.5.5) that

$$\left(G \operatorname{diag}(\zeta) + G' \operatorname{diag}(\zeta') + G'' \operatorname{diag}(\zeta'')\right) \begin{bmatrix} | \\ v_i \\ | \end{bmatrix} = [0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, 0, \ldots, 0, -1]^\top.$$

Hence, the product $\widehat{\mathbb{G}} A_5$ has the block form

$$\widehat{\mathbb{G}} A_5 = \left[\begin{array}{c|c} \operatorname{Id}_{(N-1)\times(N-1)} & \widehat{\mathbb{G}} \operatorname{Jac} y \\ \hline * & \end{array}\right], \tag{5.5.6}$$

where $*$ stands for an undetermined row vector which does not affect our argument.

Since the top $N-1$ rows of $\widehat{\mathbb{G}}$ agree with those of $\operatorname{Jac} g$ and $Dg(Y) = 0$,

$$\widehat{\mathbb{G}} \operatorname{Jac} y = [0, 0, \ldots, 0, t]^\top, \text{ where } t = \left(G_\gamma \operatorname{diag}(\zeta) + G'_\gamma \operatorname{diag}(\zeta') + G''_\gamma \operatorname{diag}(\zeta'')\right) \operatorname{Jac} y = 1,$$

the last equality being (5.5.3). This implies that $\widehat{\mathbb{G}} A_5$ is a lower-triangular matrix with ones on the main diagonal. As a consequence,

$$\det(\widehat{\mathbb{G}}) \det(A_5) = 1.$$

The remaining change-of-basis matrix at $T_u U$ reduces to $A_4 = [1]$, because $Y = Dy\left(\frac{\partial}{\partial u}\right)$.

Thus, the torsion of the complex (5.5.2) equals

$$\pm\frac{\det(A_4)\det(A_6)}{\det(A_5)\det(A_7)} = \pm\frac{1}{2}\det(\widehat{\mathbb{G}}).$$

The result obtained above can be stated as follows.

Proposition 5.5.1. *Suppose $\mathcal{T}$ is a geometric ideal triangulation of a finite-volume hyperbolic 3-manifold M with a single toroidal end and that a curve γ satisfying Assumption 1.2.1 is fixed arbitrarily. Let $\widehat{G}$, $\widehat{G}'$, $\widehat{G}''$ be the matrices of exponents in gluing equations on $\mathcal{T}$, defined in* (5.1.1). *Then the torsion of the sequence* (5.5.2) *is given by*

$$\mathbb{T}\Big(T\mathcal{GE}, \Big\{\tfrac{\partial}{\partial u}\Big\} \cup \Big\{\tfrac{\partial}{\partial z_j}\Big\}_j \cup \Big\{\tfrac{\partial}{\partial x_i}\Big\}_i \cup \{1\}, \varnothing\Big) = \pm\frac{1}{2}\det\Big(\widehat{G}\operatorname{diag}(\zeta) + \widehat{G}'\operatorname{diag}(\zeta') + \widehat{G}''\operatorname{diag}(\zeta'')\Big).$$

5.5.2 Generalization to the case of multiple ends

We now indicate how the calculation presented above can be generalized to the case of $k > 1$. We omit the obvious steps, emphasizing mostly those aspects of the calculation that differ from the one-cusped case detailed in the preceding section.

Unfortunately, in order to obtain an explicit formula for the torsion in the case of multiple ends, one has to make further assumptions on the matrix K of (1.3.2). Recall that in the calculation for $k = 1$, we chose the lift of the basis vector $1 \in \mathbb{C}$ to be $\frac{1}{2}\frac{\partial}{\partial x_N}$. Now, however, we need to find lifts of all k standard basis vectors of $\mathbb{C}^k$ to $T_1\mathbb{C}^N_\times$ and an analogous simple choice may not be possible. Therefore, for the remainder of the section we make the following assumption.

Assumption 5.5.2. We assume that the triangulation $\mathcal{T}$ has the property that for every toroidal end T_l of M, there exists an edge $e_{i(l)}$ starting and ending at T_l. In other words, $K_{l,i(l)} = 2$ for all $l \in \{1, \dots, k\}$. Furthermore, we renumber the edges in such a way that $i(l) = N - k + l$ for all l.

With the above assumption in place, the generalization is straightforward. After choosing $\frac{1}{2}\frac{\partial}{\partial x_{i(l)}}$ as the lift of the lth standard basis vector in the final term $\mathbb{C}^k$, equation (5.5.4) becomes

$$A_6 = \begin{bmatrix} \mathrm{Id}_{(N-k)\times(N-k)} & 0 \\ * & \frac{1}{2}\mathrm{Id}_{k\times k} \end{bmatrix}. \tag{5.5.7}$$

By the exactness of $T\mathcal{GE}$, we can choose the undetermined entries $*$ in such a way that the first $N - k$ columns of A_6 express a basis of $\operatorname{Im} Dg$. Choosing lifts of these vectors to $T_z\mathbb{C}^N_\times$ as $v_1, \dots, v_{N-k}$, we obtain

$$A_5 = \begin{bmatrix} | & | & & | & \\ v_1 & v_2 & \cdots & v_{N-k} & \boxed{\operatorname{Jac} y} \\ | & | & & | & \end{bmatrix},$$

where now $\operatorname{Jac} y$ has k columns. Finishing the calculation requires using (1.3.15) for general k. To this end, we define generalized matrices $\widehat{G}$, $\widehat{G}'$, $\widehat{G}''$ by

$$\widehat{G}^{\square}_{ij} = \begin{cases} G^{\square}_{ij} & \text{if } 1 \le i \le N - k, \\ G^{\square}_{\theta_l, j} & \text{if } i = i(l) \in \{N - k + 1, \dots, N\}. \end{cases} \tag{5.5.8}$$

Note that the removed edge equations are precisely those guaranteed to be redundant by Assumption 5.5.2. Now (1.3.15) yields the following generalization of (5.5.6):

$$\left(\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\right) A_5 = \begin{bmatrix} \mathrm{Id}_{(N-k)\times(N-k)} & 0 \\ * & \mathrm{Id}_{k\times k} \end{bmatrix}. \tag{5.5.9}$$

In this way, we obtain the generalized formula

$$\mathbb{T}(T\mathcal{GE}) = \pm\frac{1}{2^k}\det\Big(\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\Big). \tag{5.5.10}$$

Remark 5.5.3. We stress that Assumption 5.5.2 is needed only to produce the explicit expression given in (5.5.10), but other similar formulae could be derived even when the assumption does not hold. In general, an edge of $\mathcal{T}$ may connect two different ends of M. Choosing a set of edges spanning the entire space of end-to-edge incidences, denoted by $\mathbb{C}^k$ and forming the rightmost term of $T\mathcal{GE}$, is always possible, but may not give A_6 the lower-triangular form seen in (5.5.7).

5.6 Reduction of the conjecture

We wish to summarize the calculations of the preceding sections. Recall that in equation (5.2.7) we decomposed the adjoint non-abelian Reidemeister torsion $\mathbb{T}_{\mathrm{Ad}}(M,\gamma)$ into a product of four terms. If M has only one toroidal end ($k = 1$) and is considered with an incomplete hyperbolic structure resulting from a deformation of the unique complete structure, then we can combine equations (5.3.3) and (5.4.2) with Proposition 5.5.1 to simplify the expression for the torsion to

$$\mathbb{T}_{\mathrm{Ad}}(M,\gamma) = \pm\frac{1}{2}\det\Big(\widehat{G}\,\mathrm{diag}\,(\zeta) + \widehat{G}'\,\mathrm{diag}\,(\zeta') + \widehat{G}''\,\mathrm{diag}\,(\zeta'')\Big)\,\mathbb{T}\left(C^\bullet(X^{(1)};\mathscr{K}), \underline{c}^{(0,1)}_{\mathrm{geom}}, \underline{h}_{M_0}\right), \tag{5.6.1}$$

where the basis $\underline{h}_{M_0} \subset H^1(M;\mathscr{K}_{M_0})$ was specified in equations (5.2.6) and (5.4.3) as

$$\underline{h}_{M_0} = \eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta}. \tag{5.6.2}$$

Furthermore, since both sides of the equality (5.6.1) depend analytically on the shape parameters $z \in \mathcal{V}^+_{\mathcal{T}}$, the equality extends to the complete hyperbolic structure on M by analytic continuation. This leads us to formulate the following conjecture.

Conjecture 5.6.1 (The Reduced 1-loop Conjecture)**.** Let $X^{(1)}$ be the dual graph of a geometric ideal triangulation $\mathcal{T}$ of an open 3-manifold M with $k > 0$ toroidal ends, admitting a complete hyperbolic structure of finite volume corresponding to a positively oriented solution $z_* \in P^N$ of Thurston's gluing equations on $\mathcal{T}$. Consider any point $z \in \mathcal{V}^+_{\mathcal{T}}$ lying in the same connected component of $\mathcal{V}^+_{\mathcal{T}}$ as z_* and adopt Notational Convention 1.3.3. Then we have

$$\mathbb{T}\left(C^\bullet(X^{(1)};\mathscr{K}), \underline{c}^{(0,1)}_{\mathrm{geom}}, \eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta}\right) = \left(\zeta^f\,\zeta'^{f'}\,\zeta''^{f''}\right)^{-1}, \tag{5.6.3}$$

where (f, f', f'') is any combinatorial flattening on $\mathcal{T}$.

Note that the above reduced conjecture does not involve in any way the peripheral multicurve γ, since the balancing of cohomology bases needed for the calculation of the non-abelian torsion is accomplished by an application of Corollary 4.4.4. Thus, the necessary normalization of torsion is entirely contained in the determinant term of (5.6.1). In particular, while the original 1-loop Conjecture was stated only for $k = 1$, we now see no reason why the reduced conjecture would not hold for hyperbolic 3-manifolds with an arbitrary number of cusps.

Theorem 5.6.2. *Assume the manifold M admits a finite volume hyperbolic structure with one cusp ($k = 1$). Then*

(i) The conjectural expression (5.1.6) *for Reidemeister torsion proposed by Dimofte–Garoufalidis is never equal to zero.*

(ii) Conjecture 5.6.1 implies the 1*-loop Conjecture 5.1.5 of Dimofte–Garoufalidis.*

Proof. Equation (5.1.10) gives a symmetrized version of the conjectural expression derived by Dimofte–Garoufalidis. Part (i) follows immediately from Proposition 5.5.1 and equation (5.1.10). For Part (ii), compare (5.6.1) with (5.1.10). Q.E.D.

We remark that Part (i) of the above theorem provides what Dimofte and Garoufalidis call a "crucial ingredient" [11, p. 1258] for their perturbative definition of the higher-order terms S_n, $n \geq 2$, in the proposed asymptotic expansion mentioned at the beginning of the chapter.

5.7 Analysis of the reduced conjecture

In this section, we discuss the reduced Conjecture 5.6.1, provide further information on its properties, and speculate on possible proof strategies.

5.7.1 A further generalization of the reduced conjecture

As explained in Section 4.2, any collection of shape parameters $z \in P^N$ defines a hyperbolic structure on M_0, regardless of whether these parameters satisfy any gluing equations. Viewing this fact algebraically, for any $z \in P^N$ we can consider the local system on M_0 given by the adjoint of the $PSL_2\mathbb{C}$-representation $E(z)$ defined by z. We refer to Section 2.3.3 for a groupoid construction of this local system. Based on the above, we propose the following generalization of Conjecture 5.6.1.

Conjecture 5.7.1 (The Generic Reduced 1-loop Conjecture)**.** Let $X^{(1)}$ be the dual graph of an abstract ideal triangulation $\mathcal{T}$ of an open 3-manifold M with $k > 0$ toroidal ends. On the subspace $M_0 \subset M$, consider the adjoint local system $\mathscr{M}$ induced by the groupoid representation of equation (2.3.7) for any $z \in \mathbb{C}_\times^N$. Then the equality

$$\mathbb{T}\left(C^\bullet(X^{(1)};\mathscr{M}), \underline{c}^{(0,1)}_{\text{geom}}, \eta_2(\{\tfrac{\partial}{\partial z_1},\ldots,\tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta}\right) = \left(\zeta^f \, \zeta'^{f'} \, \zeta''^{f''}\right)^{-1} \tag{5.7.1}$$

holds on an open subset of $\mathbb{C}_\times^N$ for any combinatorial flattening (f, f', f'').

We believe that the open subset mentioned above could be at least as large as $(\mathbb{C} \setminus \{0,1\})^N$. The set $\mathbb{C} \setminus \{0,1\}$ is the space of non-degenerate ideal tetrahedra, not necessarily positively oriented [42]. It is also the common domain of holomorphicity of ζ, ζ' and ζ''.

5.7.2 Change of exponents

Dimofte and Garoufalidis' proof of invariance of the conjectural formula (5.1.6) under the change of flattening [11, Section 3.5] implies that the term $\zeta^f \zeta'^{f'} \zeta''^{f''}$ on the right-hand side of (5.6.3) does not depend on the choice of a combinatorial flattening. In other words, we have

$$\frac{\zeta^{\widetilde{f}} \zeta'^{\widetilde{f'}} \zeta''^{\widetilde{f''}}}{\zeta^f \zeta'^{f'} \zeta''^{f''}} = \pm 1 \tag{5.7.2}$$

whenever $(\widetilde{f}, \widetilde{f'}, \widetilde{f''})$ and (f, f', f'') are two combinatorial flattenings. At present, we wish to understand the behaviour of the above quotient when the exponent vectors satisfy only the first condition in the definition of a flattening, i.e., $f + f' + f'' = (1, \ldots, 1)^\top$.

We start by introducing some notation. When (x, x', x'') is a triple of vectors in $\mathbb{R}^N$, we write

$$x^\rhd \stackrel{\text{def}}{=\!=} (x_1, x_1', x_1'', x_2, x_2', x_2'', \ldots, x_N, x_N', x_N'') \in \mathbb{R}^{3N} = (\mathbb{R}^3)^N.$$

Thus, $x^\rhd$ is merely a particular rearrangement of the coordinates of the original vectors x, x' and x''. We shall denote by $\times \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ the standard cross product on $\mathbb{R}^3$; in particular we have $(1,0,0) \times (0,1,0) = (0,0,1)$. We extend this definition to a cross product on the Cartesian product of N copies of $\mathbb{R}^3$ in the obvious, 'diagonal' way:

$$\begin{aligned} &\times \colon (\mathbb{R}^3)^N \times (\mathbb{R}^3)^N \to (\mathbb{R}^3)^N, \\ &(x_1, x_2, \ldots, x_N) \times (y_1, y_2, \ldots, y_N) = (x_1 \times y_1, x_2 \times y_2, \ldots, x_N \times y_N), \qquad x_i, y_i \in \mathbb{R}^3. \end{aligned}$$

Note that the cross product of integer vectors is again an integer vector.

Proposition 5.7.2. *Suppose (f, f', f'') and $(\widetilde{f}, \widetilde{f'}, \widetilde{f''})$ are two triples of vectors in $\mathbb{Z}^N$ satisfying*

$$f_j + f_j' + f_j'' = \widetilde{f}_j + \widetilde{f}_j' + \widetilde{f}_j'' = 1 \quad \text{for every } j \in \{1, \ldots, N\}. \tag{5.7.3}$$

Then we have

$$\frac{\zeta^{\widetilde{f}} \zeta'^{\widetilde{f'}} \zeta''^{\widetilde{f''}}}{\zeta^f \zeta'^{f'} \zeta''^{f''}} = \pm z^d z'^{d'} z''^{d''}, \quad \text{where } d^\rhd = \widetilde{f}^\rhd \times f^\rhd. \tag{5.7.4}$$

Proof. This is an elementary calculation. Using Notational Convention 1.3.3, we see that the left-hand side can be rewritten as

$$\pm z^{f + f'' - \widetilde{f} - \widetilde{f''}} (1 - z)^{f' + f'' - \widetilde{f'} - \widetilde{f''}}.$$

By (5.7.3), the above expression simplifies to

$$\pm z^{\widetilde{f'} - f'} (1 - z)^{\widetilde{f} - f} = \pm \prod_{j=1}^N z_j^{\widetilde{f}_j' - f_j'} (1 - z_j)^{\widetilde{f}_j - f_j}.$$

For an arbitrarily fixed j, we shall focus on the jth term of the above product, dropping the subscript j to simplify the notation. We may write

$$\begin{aligned}\pm z^{\widetilde{f}'-f'}(1-z)^{\widetilde{f}-f} &= \pm z^{f'(\widetilde{f}'-1)+\widetilde{f}'(1-f')}(1-z)^{f(\widetilde{f}-1)+\widetilde{f}(1-f)} \\ &= \pm z^{\widetilde{f}'f''-\widetilde{f}''f'}(1-z)^{\widetilde{f}f''-\widetilde{f}''f}\left(\frac{z-1}{z}\right)^{\widetilde{f}f'-\widetilde{f}'f}.\end{aligned}$$

On the other hand, when $d^{\triangleright} = \widetilde{f}^{\triangleright} \times f^{\triangleright}$, we have

$$(d_j, d_j', d_j'') = (\widetilde{f}_j' f_j'' - \widetilde{f}_j'' f_j',\ \widetilde{f}_j'' f_j - \widetilde{f}_j f_j'',\ \widetilde{f}_j f_j' - \widetilde{f}_j' f_j).$$

Comparing the exponents and using (1.2.1) implies the desired equality. Q.E.D.

When both (f, f', f'') and $(\widetilde{f}, \widetilde{f}', \widetilde{f}'')$ are flattenings, comparing (5.7.2) and (5.7.4) leads to the conclusion that the cross product $\widetilde{f}^{\triangleright} \times f^{\triangleright}$ is a multi-index defining a monomial in the variables z, z' and z'' which is equal to ± 1 on the gluing variety. We are uncertain whether there is a geometric interpretation of this fact.

5.7.3 Computational aspects of the reduced conjecture

In principle, the torsion on the left-hand side of (5.6.3) can be calculated as the torsion of the cochain complex

$$0 \to C^0(X^{(1)}; \mathscr{K}) \xrightarrow{\delta^0} C^1(X^{(1)}; \mathscr{K}) \to 0 \qquad (5.7.5)$$

with geometric bases of the cochain spaces obtained from the basis (1.1.4) of $\mathfrak{sl}_2\mathbb{C}$ and the cohomology basis $\eta_2(\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}\}) \cup \underline{\beta}$. We summarize how the requisite geometric bases for both cochain spaces can be constructed from the geometric realisation of $X^{(1)}$ in M.

The 0-cells of $X^{(1)}$ are in a bijective correspondence with the tetrahedra of the triangulation $\mathcal{T}$. We enumerate them as $\{V_1, \ldots, V_N\}$, where we may assume $V_j \in \mathring{\Delta}_j$ for all j. Since the interior $\mathring{\Delta}_j$ of each ideal tetrahedron is a contractible open subset of M, the same is true of its intersection with M_0. Therefore,

$$C^0(X^{(1)}; \mathscr{K}) \cong \prod_{j=1}^{N} \mathscr{K}_{V_j} \cong \prod_{j=1}^{N} \mathscr{K}(\mathring{\Delta}_j),$$

which is in full analogy with the Čech cohomology constructions discussed in Section 3.6.3. In particular, a geometric basis of $C^0(X^{(1)}; \mathscr{K})$ can be obtained as follows. Let $\varphi_j \colon \mathring{\Delta}_j \to \mathbb{H}^3$ be an orientation-preserving geometric coordinate chart on the interior of the jth tetrahedron, seen as an open subset of M. Then we may equip $\mathscr{K}(\mathring{\Delta}_j)$ with the basis of pullbacks of the basis Killing vector fields $\{\mathfrak{e}, \mathfrak{h}, \mathfrak{f}\}$ on $\mathbb{H}^3$ under the geometric chart φ_j. Note that in particular we may require φ_j to map $\mathring{\Delta}_j$ to the interior of the ideal tetrahedron in $\mathbb{H}^3$ with vertices $(0, 1, \infty, z_j)$. Alternatively, we may carry out the above construction entirely in terms of germs of Killing vector fields at the point V_j. We refer to Section 3.7 for more details.

The 1-cells of $X^{(1)}$ are in a bijective correspondence with the faces of $\mathcal{T}$. Suppose that each 1-cell is oriented in an arbitrary way. We have an isomorphism

$$C^1(X^{(1)}; \mathscr{K}) \cong \prod_{F \in \{\text{faces of } \mathcal{T}\}} \mathscr{K}_{p_F},$$

where p_F is an arbitrarily chosen point inside $F \cap M_0$. As before, any germ of an orientation-preserving developing map at p_F can be used to pull back the basis $\{\mathfrak{e}, \mathfrak{h}, \mathfrak{f}\}$ to a basis of the germ space $\mathscr{K}_{p_F}$. In particular, we may require the germ to extend to a local isometry taking the interior of F to the open ideal triangle $(0, 1, \infty) \subset \mathbb{H}^3$. These are precisely the germs used in the definition of the geometric framing (Definition 2.3.5).

In the right panel of Figure 4.1.1, we see the intersection of the geometric realisation of $X^{(1)}$ in M with any tetrahedron of $\mathcal{T}$. The coboundary map δ^0 takes a germ $Y \in \mathscr{K}_{V_j}$ at the central vertex of the tetrapod to the cochain in $C^1(X^{(1)}; \mathscr{K})$ whose value is the sum of analytic continuations of Y along the four legs of the tetrapod, taken with appropriate signs depending on the chosen orientations of the 1-cells.

Using the geometric bases described above, the coboundary map δ^0 can be written down as a block matrix $\mathcal{M}(\delta^0)$ comprising $2N \times N$ lots of 3×3 blocks with complex entries. Hence, the total size of $\mathcal{M}(\delta^0)$ over $\mathbb{C}$ is $6N \times 3N$. Suppose moreover that $\mathcal{M}(H^1)$ is a matrix whose columns are the coefficient vectors, in the geometric basis of $C^1(X^{(1)}; \mathscr{K})$, of arbitrary cocycles representing the cohomology classes in $\eta_2(\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}\}) \cup \underline{\beta}$. Hence, $\mathcal{M}(H^1)$ is also a $6N \times 3N$ matrix with complex entries. By definition of torsion, we have

$$\mathbb{T}\left(C^\bullet(X^{(1)}; \mathscr{K}), \underline{c}^{(0,1)}_{\text{geom}}, \eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta}\right) = \pm \det\left[\mathcal{M}(\delta^0) \mid \mathcal{M}(H^1)\right]^{-1}. \tag{5.7.6}$$

With this notation, Conjecture 5.6.1 posits that

$$\det\left[\mathcal{M}(\delta^0) \mid \mathcal{M}(H^1)\right] \overset{?}{=} \pm \zeta^f \zeta'^{f'} \zeta''^{f''} \tag{5.7.7}$$

for any flattening (f, f', f''). Therefore, we suspect that a successful proof of Conjecture 5.6.1 should involve the upper-triangularization of the block matrix $\left[\mathcal{M}(\delta^0) \mid \mathcal{M}(H^1)\right]$.

This upper-triangularization amounts to the complex (5.7.5) admitting a filtration by nested subcomplexes; given such a filtration, a repeated application of Theorem 3.1.5 could result in the decomposition of the torsion into a product of N terms, corresponding to the ideal tetrahedra of $\mathcal{T}$. Unfortunately, it is far from clear how to obtain such a filtration, if it at all exists.

5.7.4 Good parameters for ideal tetrahedra

In this section, we continue our analysis of the reduced Conjecture 5.6.1 under the additional assumption that the triangulation $\mathcal{T}$ admits a flattening with values in $\{0, 1\}$. This assumption is satisfied for instance by the two triangulations of the figure-eight knot complement discussed in Section 1.2.

Definition 5.7.3. Let $\mathcal{T}$ be an abstract N-tetrahedron ideal triangulation of a 3-manifold with $k > 0$ toroidal ends.

(i) A *taut angle structure* on $\mathcal{T}$ is a vector $(\tau, \tau', \tau'') \in \{0, 1\}^{3N} \subset \mathbb{Z}^{3N}$ satisfying

$$\tau + \tau' + \tau'' = (1, \ldots, 1)^\top, \tag{5.7.8}$$

$$G\tau + G'\tau' + G''\tau'' = (2, \ldots, 2)^\top. \tag{5.7.9}$$

(ii) A *taut flattening* on $\mathcal{T}$ is a combinatorial flattening (f, f', f'') on $\mathcal{T}$ with values 0 and 1 only, i.e., $(f, f', f'') \in \{0, 1\}^{3N}$.

Taut angle structures were introduced by M. Lackenby in [31]. Lackenby's original definition differs from ours by a factor of π, since he thinks of taut angle structures as of generalized assignments of dihedral 'angles' to the pairs of opposite edges of the ideal tetrahedra. The reason why we work with integers is that a taut flattening is automatically a taut angle structure in our sense.

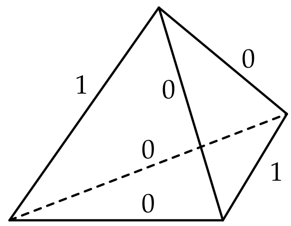


Figure 5.7.1: A tetrahedron with a taut angle structure.

Suppose (τ, τ', τ'') is a taut angle structure. Then for every $j \in \{1, \ldots, N\}$ exactly one of the numbers τ_j, τ'_j, τ''_j equals 1 and the remaining two are 0. Hence, a taut angle structure can be thought of as a choice of a pair of opposite edges (normal quadrilateral type) in every ideal tetrahedron, as depicted in Figure 5.7.1.

Proposition 5.7.4. *Let $\mathcal{T}$ be an abstract N-tetrahedron ideal triangulation of a 3-manifold with toroidal ends and let (τ, τ', τ'') be a taut angle structure on $\mathcal{T}$. Then there exists a bijection*

$$J \colon \{1, \ldots, N\} \to \{1, \ldots, N\}$$

such that for every $i \in \{1, \ldots N\}$, an edge of the tetrahedron $\Delta_{J(i)}$ with taut angle π is incident to the edge e_i of the triangulation $\mathcal{T}$.

Proof. Consider the abstract bipartite graph $G = (V, A)$ where $V = \{\Delta_1, \ldots, \Delta_N\} \sqcup \{e_1, \ldots, e_N\}$ and A is defined as follows: draw an arc between Δ_j and e_i for every edge of the tetrahedron Δ_j of taut angle π incident to e_i. Note that every tetrahedron has two (opposing) edges of taut angle π and (5.7.9) says that exactly two edges of taut angle π are glued into every edge of $\mathcal{T}$. Hence, G is a 2-regular graph and by a theorem of Frobenius, G has a perfect matching. Q.E.D.

Remark 5.7.5. Recall that the cohomology basis occurring in (5.7.6) has the form

$$\eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta} = \eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \bigcup_{i=1}^{N} \{\beta(i, \mathfrak{e}), \beta(i, \mathfrak{f})\} \subset H^1(M; \mathscr{K}_{M_0}),$$

where the cocycles $\beta(i, \mathfrak{e})$, $\beta(i, \mathfrak{f})$ satisfy (5.4.4). Note that the part of the above basis spanning the image of η_2 consists of vectors indexed by the tetrahedra of $\mathcal{T}$, whereas the vectors $\beta(i, \cdot)$ correspond to the edges of the triangulation. If (f, f', f'') is a taut flattening, then Proposition 5.7.4 implies that we can rewrite the above basis as

$$\eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta} = \bigcup_{i=1}^{N} \{\beta(i, \mathfrak{e}),\ \eta_2\Big(\frac{\partial}{\partial z_{J(i)}}\Big),\ \beta(i, \mathfrak{f})\}. \tag{5.7.10}$$

In this way, the basis is split into N subsets, each of which is associated to an edge of the ideal triangulation and the tetrahedron corresponding to that edge via the bijection of Proposition 5.7.4.

Note that the invariance under the change of flattening [11, Section 3.5] implies that if Conjecture 5.6.1 holds for one flattening, then it holds for all flattenings. Therefore, if $\mathcal{T}$ admits a taut flattening, we can simplify Conjecture 5.6.1 somewhat. For one thing, Remark 5.7.5 implies that there is a uniform way to organize the cohomology basis. In addition, we shall obtain further simplification by choosing good parameters for the ideal tetrahedra.

Recall that the monomorphism η_2 of Theorem 4.3.1 is essentially the derivative of the analytic map $E \colon P^N \to X(\pi_1(M_0))$ of equation (4.2.4). The map E assigns to every N-tuple of shape parameters $(z_1, \ldots, z_N) \in P^N$ the corresponding flat $PSL_2\mathbb{C}$-bundle on M_0. In order for this parametrization to be meaningful, we need to specify which pair of opposite edges of each tetrahedron the shape parameters $(z_1, \ldots, z_N)$ refer to. There are three possible choices in every tetrahedron and in Section 1.2 we agreed that these choices would be fixed in an arbitrary way.

A taut flattening (f, f', f'') determines a distinguished pair of opposite edges in every tetrahedron, namely the edges assigned the value 1 (see Figure 5.7.1). In what follows, we wish to study this particular choice of normal quadrilateral types. The good parameters describing the geometry of the tetrahedra of $\mathcal{T}$ will be the log-parameters associated to the distinguished edges.

Definition 5.7.6 (Good parameters). Suppose $(f, f', f'') \in \{0,1\}^{3N}$ is a taut flattening on $\mathcal{T}$. For every $j \in \{1, \ldots, N\}$, we define a map

$$Q_j \colon P \to \mathbb{C}, \qquad Q_j(z) = f_j \log z + f'_j \log z' + f''_j \log z'' = \begin{cases} \log z & \text{iff } f_j = 1, \\ \log z' & \text{iff } f'_j = 1, \\ \log z'' & \text{iff } f''_j = 1, \end{cases}$$

where log is the standard branch of the logarithm on $P = \mathbb{C}_{\mathrm{Im}>0}$. The *good parameter* on P^N given by the taut flattening (f, f', f'') is the map

$$Q \colon P^N \to \mathbb{C}^N, \quad Q(z_1, \ldots, z_N) = \big(Q_1(z_1), \ldots, Q_N(z_N)\big).$$

Note that the standard branch of the logarithm takes the upper halfplane P biholomorphically onto the horizontal strip $0 < \mathrm{Im}\, z < \pi$. Hence, Q is a biholomorphic map onto the Cartesian product of N such strips. Differentiating Q at a point $z \in \mathcal{V}^+_{\mathcal{T}}$ corresponding to a totally incomplete hyperbolic structure, we obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_z P^N & \xrightarrow{\eta_2} & H^1(M; \mathscr{K}_{M_0}) \\ & & \big\downarrow DQ & & \big\| \\ 0 & \longrightarrow & \mathbb{C}^N & \xrightarrow{\eta_Q} & H^1(M; \mathscr{K}_{M_0}) \end{array} \qquad \text{where } \eta_Q \overset{\text{def}}{=\!=} \eta_2 \circ (DQ_z)^{-1}.$$

The image of the map η_Q coincides with the image of η_2, so η_Q can be viewed as a special choice of coordinates, informed by the taut flattening (f, f', f''), on the same subspace $\mathrm{Im}\, \eta_2 \subset H^1(M; \mathscr{K}_{M_0})$.

We now need to discuss the effect of this change of parametrization on the Reidemeister torsion occurring in (5.7.6). Let $\underline{b}$ be the standard basis of $\mathbb{C}^N$. We have

$$\mathbb{T}\left(C^\bullet(X^{(1)}; \mathscr{K}),\, \underline{c}^{(0,1)}_{\mathrm{geom}},\, \eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) \cup \underline{\beta}\right) = \pm A\, \mathbb{T}\left(C^\bullet(X^{(1)}; \mathscr{K}),\, \underline{c}^{(0,1)}_{\mathrm{geom}},\, \eta_Q(\underline{b}) \cup \underline{\beta}\right),$$

where the constant A can be calculated from Theorem 3.1.4 as

$$A = \pm \det\Big[\eta_2(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) / \eta_Q(\underline{b})\Big]^{-1} = \pm \det\Big[DQ(\{\tfrac{\partial}{\partial z_1}, \ldots, \tfrac{\partial}{\partial z_N}\}) / \underline{b}\Big]^{-1} = \frac{\pm 1}{\det \mathrm{Jac}\, Q}.$$

In the above equation, $\mathrm{Jac}\, Q$ denotes the Jacobian matrix of the map Q with respect to the standard bases $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_N}\} \subset P^N$ and $\underline{b} \subset \mathbb{C}^N$. It is easy to see that (1.3.13) and the tautness of the flattening imply

$$\det \operatorname{Jac} Q = \prod_{j=1}^{N} \frac{\partial}{\partial z_j}\big(f_j \log z_j + f_j' \log z_j' + f_j'' \log z_j''\big) = \prod_{j=1}^{N}\big(f_j\zeta_j + f_j'\zeta_j' + f_j''\zeta_j''\big)$$
$$= \prod_{j=1}^{N} \zeta_j^{f_j} \zeta_j'^{f_j'} \zeta_j''^{f_j''} = \zeta^f \zeta'^{f'} \zeta''^{f''}.$$

Hence, the sought for constant A equals $(\zeta^f \zeta'^{f'} \zeta''^{f''})^{-1}$. In conclusion, the equality postulated by the reduced Conjecture 5.6.1 can be rewritten as

$$\mathbb{T}\big(C^\bullet(X^{(1)}; \mathscr{K}),\ \underline{c}^{(0,1)}_{\text{geom}},\ \eta_Q(\underline{b}) \cup \underline{\beta}\big) \overset{?}{=} \pm 1, \tag{5.7.11}$$

as long as Q is a good parameter corresponding to a taut flattening.

If the above equality is true, it can be interpreted as follows. The foregoing constructions define a basis

$$h^1_{\text{good}} \overset{\text{def}}{=\!=} \eta_Q(\underline{b}) \cup \underline{\beta} \subset H^1(M; \mathscr{K}_{M_0}),$$

where η_Q is given by a good parameter and $\underline{\beta}$ consists of cohomology classes evaluating to unit parabolic deformations of holonomies around the edges of $\mathcal{T}$ (see Section 5.4.1). Hence, equation (5.7.11) postulates that Milnor's compatibility condition holds for the short exact sequence

$$0 \to C^0(X^{(1)}; \mathscr{K}) \xrightarrow{\delta^0} C^1(X^{(1)}; \mathscr{K}) \to H^1(M; \mathscr{K}_{M_0}) \to 0.$$

In the light of Propostion 5.7.4, this statement seems plausible, although we do not know a proof.

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